CONVERGENCE ANALYSIS OF ACCELERATED ITERATIVE ALGORITHMS AND APPLICATIONS

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เรื่อง: การวิเคราะห์การลู่เข้าของอัลกอริทึมการทำซ้ำแบบเร่งและการประยุกต์
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คำสำคัญ: ทฤษฎีบทการลู่เข้า, อัตราการลู่เข้า, ปัญหาค่าต่ำสุดเชิงนูน, ปัญหาการกู้คืนภาพเบลอ, ปัญหากู้คืนสัญญาน

บทคัดย่อ

ทฤษฎีจุดตรึงมีการศึกษาในงานวิจัยต่างๆ มากมาย เนื่องจากเป็นเครื่องมือที่มีประโยชน์ในการแก้ปัญหาในสาขาต่างๆ หลากหลายสาขา เช่น วิศวกรรมศาสตร์ เศรษฐศาสตร์ เคมี และทฤษฎีเกม เป็นต้น วิธีการทำซ้ำเป็นเครื่องมือที่นิยมใช้ในการประมาณค่าจุดตรึงของการส่งแบบไม่เชิงเส้น ในคณิตศาสตร์เชิงคำนวณ เป็นเรื่องที่น่าสนใจอย่างยิ่งที่จะรู้ว่าวิธีการทำซ้ำใด จะลู่เข้าสู่คำตอบได้รวดเร็ว หรือที่เรียกกันทั่วไปว่าอัตราการลู่เข้า ดังนั้น เมื่อศึกษาขั้นตอนการทำซ้ำ เราจึงสนใจพิจารณาเกณฑ์สองข้อที่คือความเร็วและความง่าย ในวิทยานิพนธ์ฉบับนี้ได้แบ่งออกเป็นสามส่วน ส่วนแรกของวิทยานิพนธ์นี้ คือ การนำเสนอเทคนิคการนูร์แบบใหม่ที่เรียกว่าการทำซ้ำแบบ CT สำหรับการประมาณค่าจุดตรึงของฟังก์ชันต่อเนื่องบนช่วงปิด จากนั้นจะให้เงื่อนไขที่จำเป็นและเพียงพอสำหรับการลู่เข้าของการทำซ้ำแบบ CT ของฟังก์ชันต่อเนื่องบนช่วงปิด โดยเปรียบเทียบอัตราการลู่เข้าระหว่างการทำซ้ำที่สร้างขึ้นกับกระบวนการทำซ้ำแบบอื่นๆ ในงานวิจัยต่างๆ ก่อนหน้า โดยเฉพาะอย่างยิ่ง ผลลัพธ์หลักแสดงให้เห็นว่าการทำซ้ำแบบ CT สามารถลู่เข้าสู่จุดตรึงได้เร็วกว่าการทำซ้ำแบบ CP ในขั้นตอนสุดท้ายได้ตัวอย่างเชิงตัวเลขเพื่อเปรียบเทียบผลลัพธ์กับการทำซ้ำของมานน์, อิชิกาวา, นูร์, SP และ CP

ส่วนที่สองของวิทยานิพนธ์คือได้แนะนำวิธีการทำซ้ำที่รวดเร็วแบบใหม่ที่และให้ผลลัพธ์ของการลู่เข้าสำหรับการประมาณจุดตรึงของการส่งแบบไม่ขยายในปริภูมิบานาค นอกจากนี้ยังแสดงให้เห็นว่ากระบวนการทำซ้ำที่สร้างขึ้ลู่เข้าสู่จุดตรึงรวดเร็วกว่ากระบวนการทำซ้ำที่มีมาก่อนหน้านี้โดยสนับสนุนการพิสูจน์ทฤษฎีที่สร้างขึ้นด้วยตัวอย่างเชิงตัวเลข ในการประมาณค่าจุดตรึงด้วยโปรแกรม MATLAB ยิ่งไปกว่านั้นได้ประยุกต์ผลลัพธ์ที่ได้ในการหาคำตอบของปัญหาการหาค่าต่ำสุดที่จำกัด ปัญหาความเป็นไปได้ในแบบแยกส่วน และปัญหาการการกู้คืนภาพเบลอ

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ผลลัพธ์ที่ได้ในวิทยานิพนธ์ฉบับนี้ เป็นการขยาย และวางนัยทั่วไปของบางผลลัพธ์ที่เคยมีมาก่อนหน้านี้

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ABSTRACT

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry and game theory etc. Iterative methods are popular tools to approximate fixed points of nonlinear mappings. In computational mathematics, it is of vital interest to know which of the given iterative procedures converge faster to a desired solution, commonly known as the rate of convergence. Thus, when studying an iterative procedure, we should consider two criteria which are the faster and the simplify.

In this dissertation has separated by three parts. The first part of this dissertation is to propose a novel Noor iteration technique, called the CT–iteration for approximating a fixed point of continuous functions on closed interval. Then, a necessary and sufficient condition for the convergence of the CT–iteration of continuous functions on closed interval is established. We also compare the rate of convergence between the proposed iteration and some other iteration processes in the literature. Specifically, our main result shows that CT–iteration converges faster than CP–iteration to the fixed point. We finally give numerical examples to compare the result with Mann, Ishikawa, Noor, SP and CP iterations.

The second part of dissertation is to introduce a new faster iteration scheme and establish convergence results for approximation of fixed points of nonexpansive mappings in the framework of Banach spaces. Further, we show that our iteration process is faster than a number of existing iteration processes. We support our analytic proof by numerical examples in which we approximate the fixed point by a computer using MATLAB program. Furthermore, we apply our results to find solutions of constrained minimization problems, split feasibility problems and image deblurring problems.

Third, using sunny nonexpansive retractions, which are different from the metric projection in Banach spaces, a new type of study regarding the iterative methods in view of two quasi–nonexpansive nonself mappings is presented. We also give the convergence analysis for the proposed method in the background of uniformly convex Banach spaces. Moreover, we apply our results to find solutions of common zeros of accretive operators, convexly constrained least square problems, and convex minimization problems. Furthermore, we also discuss novel applications of these methods to differential problems, image deblurring, and signal recovering problems.

The results obtained in this dissertation extend and generalize some results in the literature.

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CHAPTER I

INTRODUCTION

Fixed point theory takes a large amount of literature, since it provides useful tools to solve many problems that have applications in different fields like engineering, economics, chemistry and game theory etc. Iterative methods are popular tools to approximate fixed points of nonlinear mappings. In computational mathematics, it is of vital interest to know which of the given iterative procedures converge faster to a desired solution, commonly known as the rate of convergence. Thus, when studying an iterative procedure, we should consider two criteria which are the faster and the simplify. In this direction, some of notable studies were conducted by Mann, Ishikawa, Noor, Phuengrattana and Suantai, Cholamjiak and Pholasa (see [16, 24, 34, 35, 41]). In addition, the fixed point mappings were studied as much as studies on the iterative methods. Different varieties of these mappings are available in the literature. The well known of them, are contraction mappings, nonexpansive mappings and Lipschitzian mappings, and these are the continuous ones. Therefore, in this study, we handle the general mapping which is a class of continuous mapping.

Let C be a closed interval on the real line and let $f : C \to C$ be a continuous function. A point $p \in C$ is called a *fixed point* of f if $f(p) = p$.

Now, we will consider some of these schemes related to this work.

Mann [34] introduced Mann iteration, which generates a sequence $\{u_n\}$ as follows :

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n f(u_n) \tag{1.0.1}$$

for all $n \geq 1$, where $\alpha_n \in [0, 1]$. Such an iteration process is known as *Mann* *iteration.* In 1991, Borwein and Borwein [10] proved the convergence theorem for a continuous function on the closed and bounded interval in the real line by using iteration (1.0.1).

Another classical iteration process was introduced by Ishikawa [24] which is formulated as follows:

$$\begin{aligned} t_n &= (1-\beta_n)s_n + \beta_n f(s_n), \\ s_{n+1} &= (1-\alpha_n)s_n + \alpha_n f(t_n) \end{aligned} \tag{1.0.2}$$

for all $n \geq 1$, where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1]. Such iterative method is called *Ishikawa iteration.* In 2006, Qing and Qihou [45] proved the convergence theorem of the sequence generated by iteration (1.0.2) for a continuous function on the closed interval in the real line (see also [44]).

In 2000, Noor [35] defined the following iterative scheme by $l_1 \in C$ and

$$\begin{aligned} m_n &= (1-\mu_n)l_n + \mu_n f(l_n), \\ v_n &= (1-\beta_n)l_n + \beta_n f(m_n), \\ l_{n+1} &= (1-\alpha_n)l_n + \alpha_n f(v_n) \end{aligned} \tag{1.0.3}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1], which is called *Noor iteration* [35] for continuous functions on an arbitrary interval in the real line.

Clearly, the Mann and Ishikawa iteration processes are special cases of the Noor iteration process. Because of its simplicity, the method (1.0.3) has been widely utilized to solve the fixed point problem, and as a result, it has been enhanced by many works, as seen in [27, 36, 37, 47].

In 1976, Rhoades [49] proved the convergence of the Mann and Ishikawa iterations for the class of continuous and nondecreasing functions on unit closed interval. After that in 1991, Borwein and Borwein [10] obtained the convergence result to Mann iteration for continuous functions on a bounded closed interval. Qing and Qihou [45] extended results in [10] to an arbitrary interval and to Ishikawa iteration and presented a necessary and sufficient condition for the convergence of Ishikawa iteration of continuous functions on an arbitrary interval (see also [44]). There are many articles have been published on the iterative methods using for approximation of fixed points of nonlinear mappings, see for instance ([10, 24, 34, 35, 44, 45, 49]). However, there are only a few articles concerning comparison of those iterative methods in order to establish which one converges faster. As far as we know, there are two ways for comparison of the rate of convergence. The first one was introduced by Berinde [8]. He used this idea to compare the rate of convergence of Picard and Mann iterations for a class of Zamfirescu operators in arbitrary Banach spaces. Popescu [43] also used this concept to compare the rate of convergence of Picard and Mann iterations for a class of quasi–contractive operators. It was shown in [57] that the Mann and Ishikawa iterations are equivalent for the class of Zamfirescu operators. In 2006, Babu and Prasad [6] showed that the Mann iteration converges faster than the Ishikawa iteration for this class of operators. Two years later, Qing and Rhoades [46] provided an example to show that the claim of Babu and Prasad [6] is false. However, this concept is not suitable or cannot be applied to a class of continuous self–mappings defined on a closed interval. In order to compare the rate of convergence of continuous self-mappings defined on a closed interval, Rhoades [49] introduced the other concept which is slightly different from that of Berinde to compare iterative methods which one converges faster as follow.

Definition 1.0.1 ([49]) Let C be a closed interval on the real line and let $f : C \to C$ be a continuous mapping. Suppose that $\{x_n\}$ and $\{w_n\}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}$ is said to converge faster than $\{w_n\}$ if

$$|x_n - p| \leq |w_n - p|$$

for all $n \geq 1$.

Phuengrattana and Suantai [41] introduced and studied the SP–iteration as follows: $h_1 \in C$ and

$$\begin{aligned} e_n &= (1-\mu_n)h_n + \mu_n f(h_n), \\ d_n &= (1-\beta_n)e_n + \beta_n f(e_n), \\ h_{n+1} &= (1-\alpha_n)d_n + \alpha_n f(d_n) \end{aligned} \tag{1.0.4}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\mu_n\}$ are sequences in [0,1]. They showed that (1.0.4) converges to a fixed point of f. Moreover, the rate of convergence is better than those of Mann (1.0.1), Ishikawa (1.0.2) and Noor (1.0.3) in the sense of Rhoades [49].

Clearly Mann iteration is special cases of SP–iteration. Some interesting results concerning fixed point theory of continuous functions can be found in [20].

Recently, by combining the SP-iteration and Noor iteration, Cholamjiak and Pholasa [16] proposed the CP–iteration as follows: $w_1 \in C$ and

$$\begin{aligned} r_n &= (1-\mu_n)w_n + \mu_n f(w_n), \\ q_n &= (1-\tau_n-\beta_n)w_n + \tau_n r_n + \beta_n f(r_n), \\ w_{n+1} &= (1-\gamma_n-\alpha_n)r_n + \gamma_n q_n + \alpha_n f(q_n) \end{aligned} \tag{1.0.5}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\mu_n\}$, $\{\tau_n\}$ and $\{\gamma_n\}$ are sequences in $[0, 1]$. They proved some convergence theorems of such iterations for continuous functions on an arbitrary interval. Also, they compared the rate of convergence of Mann, Ishikawa, Noor and CP iterations by numerical examples and concluded that CP–iteration converges faster than all of them.

Inspired and motivated by these facts, we introduce and study a new accelerated iteration process for solving a fixed point problem for continuous function on an arbitrary interval in the real line. The scheme is defined as follows.

Let C be a closed interval on the real line and $f : C \to C$ given mapping. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned} \tag{1.0.6}$$

where, $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ are appropriate real sequences in $[0, 1]$. The iterative scheme (1.0.6) is called the CT–iteration for continuous functions.

Once the existence of a fixed point of some mapping is established, then to find the value of that fixed point is not an easy task, that is why we use iterative processes for computing them. By time, many iterative processes have been developed and it is impossible to cover them all.

Numerical reckoning fixed points for nonlinear operators is nowadays an active research direction of nonlinear analysis. This because they found applications to variational inequalities, equilibrium problems, computer simulation, image encoding and much more. Classical iterations such as Picard, Mann and Ishikawa represent pioneers research work in this regard; please, see Mann [34] and Ishikawa [25]. Nowadays, this research direction is developed by Agarwal et al. [2] and Noor [35]. Speed of convergence play important role for an iteration process to be preferred on another iteration process. In [50], Rhoades mentioned that the Mann iteration process for decreasing function converges faster than the

Ishikawa iteration process and for increasing function the Ishikawa iteration process is better than the Mann iteration process. Also the Mann iteration process appears to be independent of the initial guess (see also [51]). In [2], the authors claimed that Agarwal iteration process converge at a rate same as that of the Picard iteration process and faster than the Mann iteration process for contraction mappings.

Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E. Throughout this work, $\mathbb{N}$ denotes the set of all positive integers $T : C \to C$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$ and for all $n \in \mathbb{N}$. For arbitrary chosen $x_1 \in C$, construct a sequence $\{x_n\}$, where x_n is defined iteratively for each positive integer $n \geq 1$ by:

$$x_{n+1} = Tx_n, \tag{1.0.7}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \tag{1.0.8}$$

$$\begin{cases} x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n, \\ y_n = (1 - \beta_n)x_n + \beta_n Tx_n. \end{cases} \tag{1.0.9}$$

The sequences $\{x_n\}$ generated by (1.0.7), (1.0.8) and (1.0.9) are called Picard, Mann [34] and Ishikawa [24] iteration sequences respectively.

In 1955, Krasnoselskii [32] showed that the Picard iteration scheme (1.0.7) for a nonexpansive mapping T may fail to converge to fixed point of T even if T has a unique fixed point, but the Mann sequence (1.0.8) for $\alpha_n = \frac{1}{2}, \forall n \geq 1$ converges strongly to the fixed point of T.

Mann and Ishikawa iteration methods have been studied by several authors for approximation fixed points of nonexpansive mappings, see, e.g., [3, 4, 25, 29, 30, 48, 55, 58, 61, 63, 65, 69].

In 2000, Noor [35] defined the following iterative scheme, by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n \end{cases} \tag{1.0.10}$$

for all $n \geq 1$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in (0,1).

Recently, Agarwal et al. [2] introduced the following iteration process. For arbitrary chosen $x_1 \in C$ construct a sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T x_n, \qquad n \in \mathbb{N}, \end{cases} \tag{1.0.11}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in (0,1). They showed that this process converges at a rate that is the same as that of the Picard iteration (1.0.7) and faster than the Mann iteration (1.0.8) for contractions mapping.

Motivated by the previous ones, we introduce a new faster iteration process for numerical reckoning fixed points of nonexpansive mappings, where the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n T y_n, \\ y_n = (1-\beta_n)Tx_n + \beta_n T z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n T x_n, \end{cases} \tag{1.0.12}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in (0,1).

There is also another interesting problem related to zero point problem. Assume that $\mathcal{E}$ is a real Banach space. Suppose that $\mathcal{A} : \mathcal{E} \to \mathcal{E}$ is an operator and $\mathcal{B} : \mathcal{E} \to 2^{\mathcal{E}}$ is a set–valued operator. We study the following zero point problem: find an element $u \in \mathcal{E}$ such that

$$0 \in \mathcal{A}u + \mathcal{B}u. \tag{1.0.13}$$

This topic is generally known to consist of specific instances, convex programming variational inequalities, split feasibility problem and minimization problem ([14, 17, 26, 28, 31, 39, 58]) which have applications in data science, image recovery and signal processing [31, 39] can be designed a mathematical model like this from (1.0.13).

Many mathematicians have been interested in analyzing fixed points using some iterative methods in recent years. With the considerable improvements in fixed point theory in the last several years, iterative concepts have emerged as a topic of increasing interest. Iteration qualities involving types of sequences and types of operators have not been thoroughly investigated and are currently being debated. Because of its considerable usefulness in fixed point theory and its implementations, the idea of operators has populated a prominent part in present scientific studies employing algorithmic approaches. A number of studies (see [9]) have observed fixed points using iterative techniques. It is worth noting that the type of operators used in fixed point studies becomes important.

Let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{E}$ and $\mathcal{S} : \mathcal{K} \to \mathcal{K}$ is the operator with at least one fixed point. Many researchers suggested various strategies for finding an approximate solution in order to analyse a fixed point. Picard iteration process [40] is one of the most widely utilized iterative processes, was represented like (1.0.7).

Other iterative methods for modifying the Picard iteration method have also been efficiently researched, such as Mann iteration (1.0.8), and Ishikawa iteration (1.0.9).

Agarwal, O'Regan, and Sahu [2] presented the S–iteration method in the Banach space is defined by:

$$u_{n+1} = (1-\eta_n)\mathcal{S}u_n + \eta_n\mathcal{S}((1-\vartheta_n)u_n + \vartheta_n\mathcal{S}u_n),\ n \geq 1,$$

where $\{\eta_n\}$ and $\{\vartheta_n\}$ are sequences in $(0,1)$. They showed that this iteration method is independent of the process of iteration Mann and Ishikawa and converges faster than both.

Abbas and Nazir [1] proposed a new iteration technique for approximating fixed points of nonexpansive mappings in uniformly convex Banach space in 2014. The sequence $\{x_n\}$ in this system model, beginning at the initial prediction $u = u_1 \in C$, is given by:

$$\begin{aligned} w_n &= (1-\zeta_n)u_n + \zeta_n\mathcal{S}u_n, \\ z_n &= (1-\vartheta_n)\mathcal{S}u_n + \vartheta_n\mathcal{S}w_n, \\ u_{n+1} &= (1-\eta_n)\mathcal{S}z_n + \eta_n\mathcal{S}w_n, \quad n \geq 1, \end{aligned}$$

where $\{\eta_n\}$, $\{\vartheta_n\}$ and $\{\zeta_n\}$ are sequences in $(0,1)$. According to the researchers, this strategy converges to a fixed point of contraction mapping faster than all Picard, Mann, and Agarwal iteration processes. Some of the other well–known three–step iteration processes are Noor [35], Ullah and Arshad [64] (AK-iteration), Sahu et al. [53] and Thakur et al. [62] and Phuengrattana and Suantai [41] (SP-iteration).

Signal processing and numerical optimization are two separate science–based topics that have interacted. Compressed sensing (CS) [14] is perhaps the most convincing example of the two fields interacting. There have been several analyses related to these algorithms and their applicability in signal processing [17, 19, 26, 58].

The first purpose of this dissertation is to give a necessary and sufficient condition for the strong convergence of the CT–iteration of continuous functions on an arbitrary interval and improve the rate of convergence compared to previous work. Specifically, our main result shows that CT–iteration converges faster than CP–iteration to the fixed point. Numerical examples are also presented to compare the result with Mann, Ishikawa, Noor, SP and CP iterations. Consequently, we have that CT–iteration converges faster than the other schemes in the same category.

The second purpose of this dissertation is to prove convergence results for nonexpansive mappings using the iteration (1.0.12). We also prove that the iteration (1.0.12) converges faster than Picard, Mann, Ishikawa, Noor and Agarwal et al. iteration processes for contractive mappings in the sense of Berinde [8]. We also present numerical examples to compare the convergence of (1.0.12) with Picard, Mann, Ishikawa, Noor and Agarwal et al. iterations. Moreover, we apply our results to find solutions of constrained minimization problems, split feasibility problems and image deblurring problems.

The third purpose of this dissertation is to construct a new iteration process for calculating common solutions of the common fixed point problems and apply our results for solving the problem in (1.0.13). Furthermore, we find common solutions of convexly constrained least square problems, convex minimization problems and applied to differential problems, image restoration and signal processing.

The dissertation is divided into 4 chapters. Chapter 1 is an introduction to the research problems. Chapter 2 deals with basic concepts and preliminaries and give some useful results that will be used in later chapters. Chapter 3 is the main results of this research with divided into 3 sections as follows:

(1) Novel Noor iterations technique for solving nonlinear equations.

(2) Numerical reckoning fixed points for nonexpansive mappings via a faster iteration process and its application to constrained minimization problems, split feasibility problems and image deblurring problems .

(3) New iterative methods for nonlinear operators as concerns convex programming applicable in differential problems, image deblurring and signal recovering problems.

Chapter 4 summarizes all the theorems in this dissertation.

CHAPTER II

PRELIMINARIES

2.1 Metric Spaces, Linear spaces, Normed spaces and Banach spaces

We study functions defined on the real line $\mathbb{R}$. Every pair of points have available a distance function,call d, denote by $d(x, y) = |x - y|$.

Definition 2.1.1 [33] A **metric space** is a pair (X, d), where X is a set and d is a metric on X (or distance function on X), that, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:

(1) $d(x, y) \geq 0$,

(2) $d(x, y) = 0$ if and only if $x = y$,

(3) $d(x, y) = d(y, x)$ (symmetry),

(4) $d(x, y) \leq d(x, z) + d(z, y)$(triangle inequality).

Example 2.1.2 In real line $\mathbb{R}$, define

$$d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}$. Then $(\mathbb{R}, d)$ is a metric space.

Example 2.1.3 In euclidean plane $\mathbb{R}^2$, define

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2},$$

where $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a metric space.

Example 2.1.4 Let X be the set of all bounded sequences of complex numbers; that is every element of X is a complex sequence

$$x = (\xi_1, \xi_2, ...)$$

such that $|\xi_j| \leq c_x$ for all $j = 1, 2, ...$ and c_x is a real number which may depend on x, but does not depend on j and define

$$d(x, y) = \sup_{j\in\mathbb{N}} |\xi_j - \eta_j|,$$

where $y = (\eta_j) \in X$ and $\mathbb{N} = 1, 2, ...$ Then (X, d) is a metric space.

We first consider important types of subsets a given metric space (X, d).

Definition 2.1.5 [33] Given a point $x_0 \in X$ and a real number $r > 0$, we define three types of sets:

(a) $B(x_0; r) = x \in X|d(x, x_0) < r$ (Open ball),

(b) $\tilde{B}(x_0; r) = x \in X|d(x, x_0) \leq r$ (Closed ball),

(c) $S(x_0; r) = x \in X|d(x, x_0) = r$ (Sphere).

In all three cases, x_0 is called the center and r the radius.

Definition 2.1.6 [33] Let (X, d) be a metric space. A subset $U \subseteq X$ is open if for every $x \in X$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A set U is closed if its complement, $X \setminus U$, is open.

Definition 2.1.7 [33] Let $X = (X, d)$ and $Y = (Y, \tilde{d})$ be metric spaces. A mapping $T : X \to Y$ is said to be continuous at a point $x_0 \in X$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\tilde{d}(Tx, Tx_0) < \epsilon \text{ for all } x \text{ satisfying } d(x.x_0) < \delta.$$

T is said to be continuous if it continuous at every point of X.

Definition 2.1.8 [33] A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \quad \text{or, simple} \quad x_n \to x$$

we say that $\{x_n\}$ converges to x. If $\{x_n\}$ is not convergent, it is said to be divergent.

We call a nonempty subset $M \subset X$ a bounded set if its diameter

$$\delta(M) = \sup_{x,y\in M} d(x,y)$$

is finite. And we call a sequence $\{x_n\}$ in X a bounded sequence if the corresponding point set is a bounded subset of X.

Definition 2.1.9 [33] A sequence $\{x_k\}$ in a metric space X is said to be *Cauchy* if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_j, x_k) < \epsilon$ for all $j, k > N$.

The space X is said to be complete if every Cauchy sequence in X converges.

Theorem 2.1.10 [33] *The real line and the complex plane are complete metric spaces.*

Theorem 2.1.11 [33] *A mapping $T : X \to Y$ of a metric space (X, d) into a metric space $(Y, \tilde{d})$ is a continuous at a point $x_0 \in X$ if and only if*

$$x_n \to x_0 \ \textit{ implies } \ Tx_n \to Tx_0.$$

Particularly useful and important metric spaces are obtained if we take a vector space and define on it a metric by means of norm.

Definition 2.1.12 [33] A **linear space** or **vector space** X over $\mathbb{R}$ is a set X endowed with structure by the prescription of

(i) an operation $X \times X \to X$ called the *addition* in X, denoted by $+$;

(ii) an operation $\mathbb{R} \times X \to X$ called the *scalar multiplication* in X, denoted by (α, x) to αx in X.

The set X satisfying the following properties for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{R}$:

(1) $x + y = y + x$;

(2) $(x + y) + z = x + (y + z)$;

(3) there exists an element $0 \in X$ called the *zero vector* of X such that $x+0 = x$ for all $x \in X$;

(4) for every element $x \in X$, there exists an element $-x \in X$ called the *additive inverse* or the *negative* of x such that $x + (-x) = 0$;

(5) $\alpha(x + y) = \alpha x + \alpha y$;

(6) $(\alpha + \beta)x = \alpha x + \beta x$;

(7) $(\alpha\beta)x = \alpha(\beta x)$;

(8) $1 \cdot x = x$.

Example 2.1.13 Consider the set

$$P(v_1, v_2, \ldots, v_n) = \left\{ \sum_{i=1}^{n} a_i v_i : a_1, a_2, \ldots, a_n \in \mathbb{R} \right\}.$$

For any $x = a_1v_1 + \ldots + a_nv_n, y = b_1v_1 + \ldots + b_nv_n \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we define their *addition* $x + y$ and their *scalar multiplication* αx by

$$x + y = (a_1 + b_1)v_1 + \cdots + (a_n + b_n)v_n \quad \text{and} \quad \alpha x = \alpha a_1 v_1 + \cdots + \alpha a_n v_n.$$

Then $x + y \in P(v_1, v_2, \ldots, v_n)$ and $\alpha x \in P(v_1, v_2, \ldots, v_n)$. Further we have that these operations satisfy (1)-(8). Hence $P(v_1, v_2, \ldots, v_n)$ is a linear space with these operations.

Example 2.1.14 Consider the *Euclidean space* or *n-tuple space*

$$X = \mathbb{R}^n = \{x = (a_1, a_2, \ldots, a_n) : a_i \in \mathbb{R}\}.$$

For any $x = (a_1, a_2, \ldots, a_n), y = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$, we define their *addition* $x + y$ and their *scalar multiplication* αx by

$$x + y = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n) \quad \text{and} \quad \alpha x = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n).$$

Then $x + y \in \mathbb{R}^n$ and $\alpha x \in \mathbb{R}^n$. Further we have that these operations satisfy (1)-(8). Hence $X = \mathbb{R}^n$ is a linear space with these operations.

Definition 2.1.15 [33] A **normed space** is a pair $(X, \|\cdot\|)$ where X is a linear space over $\mathbb{R}$ and $\|\cdot\| : X \to \mathbb{R}$ is a function satisfying the following properties:

(1) $\|x\| \geq 0$ for all $x \in X$ (nonnegative);

(2) $\|x\| = 0$ if and only if $x = 0$ (strictly positive);

(3) $\|\alpha x\| = |\alpha|\|x\|$ for all scalars $\alpha \in \mathbb{R}$ and each $x \in X$ (homogeneous);

(4) $\|x + y\| \leq \|x\| + \|y\|$ for each $x, y \in X$ (triangle inequality).

The function $\|\cdot\|$ is called a *norm* on X. The norm space just defined is denoted by $(X, \|\cdot\|)$.

A norm on X defines a metric d on X which is given by

$$d(x, y) = \|x - y\|, \quad (x, y \in X)$$

and is called the metric induced by the norm.

The norm is continuous, that is $x \to \|x\|$ is a continuous mapping of $(X, \|\cdot\|)$ into $\mathbb{R}$.

Example 2.1.16 $\mathbb{R}^k$ is a normed space with the following norms:

$$\|x\|_1 = \sum_{i=1}^{k} |x_i| \text{ for all } x = (x_1, x_2, .., x_k) \in \mathbb{R}^k;$$

$$\|x\|_p = \Big(\sum_{i=1}^{k} |x_i|^p\Big)^{1/p} \text{ for all } x = (x_1, x_2, .., x_k) \in \mathbb{R}^k \text{ and } p \in (1, \infty);$$

$$\|x\|_\infty = \max_{1 \le i \le k} |x_i| \text{ for all } x = (x_1, x_2, .., x_k) \in \mathbb{R}^k.$$

Example 2.1.17 Let $X = l_1$, the linear space whose elements consist of all absolutely convergent sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),

$$l_1 = \{x : x = (x_1, x_2, ..., x_i, ...) \text{ and } \sum_{i=1}^{\infty} |x_i| < \infty\}.$$

Then l_1 is a normed space with the norm defined by $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Example 2.1.18 Let $X = l_p$ $(1 < p < \infty)$ be the linear space whose elements consist of all p–summable sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),

$$l_p = \{x : x = (x_1, x_2, ..., x_i, ...) \text{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

Then l_p is a normed space with the norm defined by $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$.

Example 2.1.19 Let $X = l_\infty$, the linear space whose elements consist of all bounded sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ *or* $\mathbb{C}$),

$$l_\infty = \{x : x = (x_1, x_2, ..., x_i, ...) \text{ and } \{x_i\}_{i=1}^{\infty} \text{ is bounded}\}.$$

Then l_∞ is a normed space with the norm defined by $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

Theorem 2.1.20 [33] *The norm associated with an inner product satisfies the thiangle inequality:*

$$\|x+y\| \leq \|x\| + \|y\|.$$

Theorem 2.1.21 [60] *For any inner product space X, the following holds:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2.$$

Definition 2.1.22 [33] Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. The operator T is said to be *bounded* if there is a real number $c > 0$ such that for all $x \in X$,

$$\|Tx\| \leq c\|x\|.$$

Definition 2.1.23 [33] A sequence x_n in a normed space X is said to be strongly convergent (or convergent in the norm) if there is an $x \in X$ such that

$$\lim_{n\to\infty} \|x_n - x\|.$$

This is written

$$\lim_{n\to\infty} x_n = x$$

or simply

$$x_n \to x.$$

x is called the strong limit of x_n, and we say that x_n converges strongly to x.

Weak convergence is defined in terms of bounded linear functionals on X as follows.

Definition 2.1.24 [33] A sequence x_n in a normed space X is said to be weakly convergent if there is an $x \in X$ such that for every $f \in X'$,

$$\lim_{n\to\infty} f(x_n) = f(x).$$

This is written

$$x_n \rightharpoonup x.$$

x is called the weak limit of x_n, and we say that x_n converges weakly to x.

Definition 2.1.25 [33] A sequence $\{x_n\}_{n=0}^{\infty}$ in a normed space X is said to be *bounded* if there exists a positive number M such that $\|x_n\| \leq M$ for all $n \geq 1$.

Definition 2.1.26 [60] A sequence $\{x_n\}_{n=0}^{\infty}$ in a normed space X is said to be a *Cauchy sequence* if there exists a sequence $\{\alpha_n\}_{n=0}^{\infty} \subset [0, \infty)$ such that $\|x_n - x_l\| \leq \alpha_n$ for all $l \geq n \geq 1$ and $\lim_{n\to\infty} \alpha_n = 0$.

Definition 2.1.27 [33] A normed space is said to be *complete* if every Cauchy sequence is convergent.

A **Banach space** is a complete normed space (complete in the metric defined by the norm.). Let $\mathcal{E}$ be a real Banach space with norm $\|\cdot\|$ and $\mathcal{E}^*$ be its dual. The value of $f \in \mathcal{E}^*$ at $u \in \mathcal{E}$ is denoted by $\langle u, f\rangle$. We denote by $\mathcal{B}_\lambda[v]$ the closed ball with the center at v and radius λ:

$$\mathcal{B}_\lambda[v] = \{u \in \mathcal{E} : \|v - u\| \leq \lambda\}.$$

Definition 2.1.28 [59] A Banach space $\mathcal{E}$ is said to be uniformly convex if for any χ, $0 < \chi \leq 2$, the inequalities $\|u\| \leq 1$, $\|v\| \leq 1$ and $\|u - v\| \geq \chi$ imply there exists $\delta = \delta(\chi) > 0$ such that $\frac{1}{2}\|u + v\| \leq 1 - \delta$.

Definition 2.1.29 [59] Let $\mathcal{E}$ be a Banach space and $S_{\mathcal{E}} = \{x \in \mathcal{E} : \|x\| = 1\}$ unit sphere on $\mathcal{E}$. For all $\lambda \in (0, 1)$, and $x, y \in S_{\mathcal{E}}$ with $x \neq y$, if $\|(1-\lambda)x + \lambda y\| < 1$, then $\mathcal{E}$ is called strictly convex.

If $\mathcal{E}$ is a strictly convex Banach space and $\|x\| = \|y\| = \|\alpha x + (1-\alpha)y\|$ for $x, y \in \mathcal{E}$ and $\alpha \in (0,1)$, then $x = y$.

Definition 2.1.30 [59] The space $\mathcal{E}$ is said to be smooth if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1.1}$$

exists for each x and y in $S_{\mathcal{E}}$. In this case, the norm of $\mathcal{E}$ is called Gateaux differentiable.

For all $y \in S_{\mathcal{E}}$, if the limit (2.1.1) is attained uniformly for $x \in S_{\mathcal{E}}$, then the norm is said to be uniformly Gateaux differentiable or Frechet differentiable.

2.2 Inner product spaces and Hilbert spaces

Definition 2.2.1 [33] An **inner product** space is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping of $X \times X$ into the scalar field $\mathbb{K}$ of X; that is, with every pair of vectors x and y there is associated a scalar which is written by $\langle x, y\rangle$ and called the *inner product* of x and y, such that for all vectors x, y, z and scalars α we have

(IP1) $\langle x, x\rangle \geq 0$;

(IP2) $\langle x, x\rangle = 0 \Leftrightarrow x = 0$;

(IP3) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$;

(IP4) $\langle x, y\rangle = \overline{\langle y, x\rangle}$;

(IP5) $\langle x+y, z\rangle = \langle x, z\rangle + \langle z, y\rangle$.

Example 2.2.2 The function $\langle \cdot,\cdot\rangle : \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$ defined by

$$\langle x, y\rangle = \sum_{i=1}^{k} x_i y_i \text{ for all } x = (x_1, x_2, ..., x_k),\ y = (y_1, y_2, ..., y_k) \in \mathbb{R}^k \tag{2.2.1}$$

is an inner product on $\mathbb{R}^k$. In this case $\mathbb{R}^k$ with this inner product is called real Euclidean k-space.

Definition 2.2.3 [33] An inner product space which is complete with respect to the induced norm is called a **Hilbert space**.

Example 2.2.4 The Euclidean space $\mathbb{R}^k$ is a Hilbert space with inner product defined by

$$\langle x, y\rangle = x^1y^1 + x^2y^2 + ... + x^ky^k$$

where $x = (x^1, x^2, ..., x^k),\ y = (y^1, y^2, ..., y^k) \in \mathbb{R}^k$.

2.3 Convex sets and convex functions

Definition 2.3.1 [59] Let C be a subset of a linear space X. Then C is said to be **convex** if $(1-\lambda)x + \lambda y \in C$ for all $x, y \in C$ and all scalar $\lambda \in [0, 1]$.

Example 2.3.2 1. Every subspace of vector space is convex.

2. $\overline{B}(x; r) = \{x : \|x\| \leq r\}$ is convex.

3. $[0, 1]^K = [1, 0] \times [1, 0] \times ... \times [1, 0]$ is convex in $\mathbb{R}^k$.

Proposition 2.3.3 [59] *Let C be a subset of a linear space X. Then C is convex if and only if $\lambda_1x_1 + \lambda_2x_2 + ... + \lambda_kx_k \in C$ for any finite set $\{x_1, x_2, ..., x_k\} \subseteq C$ and scalars $\lambda_i \geq 0$ with $\lambda_1 + \lambda_2 + ... + \lambda_k = 1$.*

Definition 2.3.4 [59] Let X be a linear space and $f : X \to (-\infty, \infty]$ be a function. Then f is said to be **convex** if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in X$ and $\lambda \in [0, 1]$.

Example 2.3.5 1. $f(x) = |x|^p$ where $p \geq 1$ is a convex function in $\mathbb{R}$.

2. $f(x) = x^3 - x^2$ is a convex function in $[\frac{1}{3}, \infty)$.

3. $f(x) = x \log x$ is a convex function in $\mathbb{R}^+$.

Definition 2.3.6 [60] A function $f : H \to \mathbb{R}$ is said to be *lower semicontinuous* on H if for any $\lambda \in \mathbb{R}$, the set

$$\{x \in H : f(x) \leq \lambda\} \quad \text{is closed.}$$

$f : H \to \mathbb{R}$ is also said to be *convex* if for any $x, y \in H$ and for any $\alpha \in (0,1)$, there holds

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y).$$

Definition 2.3.7 [60] Let $f : H \to \mathbb{R}$ be a function. For a sequence $\{x_n\}_{n=0}^{\infty} \subset H$ the *limit inferior* of $\{f(x_n)\}_{n=0}^{\infty}$ in $[-\infty, +\infty]$ is

$$\liminf_{n\to\infty} f(x_n) := \sup_{n\geq 1} \inf_{n\leq N} f(x_N)$$

and its *limit superior* in $[-\infty, +\infty]$ is

$$\limsup_{n\to\infty} f(x_n) := \inf_{n\geq 1} \sup_{n\leq N} f(x_N).$$

Definition 2.3.8 [60] A function $h : H \to \mathbb{R} \cup \{+\infty\}$ be a proper, lower semi-continuous (l.s.c.), and convex function. The subdifferential of h at x is defined by

$$\partial h(x) = \{v \in H : \langle v, y-x\rangle + h(x) \leq h(y), y \in H\}.$$

2.4 Fixed points of continuous, nonexpansive and sunny nonexpansive mappings

Let C be a closed interval on the real line and let $f : C \to C$ be a continuous function. A point $p \in C$ is called a *fixed point* of f if $f(p) = p$.

Definition 2.4.1 [67] Let C be subset of a Banach space X. A mapping $T : C \to C$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. The set of all fixed points of T is denoted by $F(T) = \{x \in C | x = Tx\}$.

Definition 2.4.2 [60] Let (X, d) be a metric space. Then a map $T : X \to X$ is said to be *contractive* if there exists $k \in [0, 1)$ such that

$$d(T(x), T(y)) \leq kd(x, y)$$

for all $x, y \in X$.

Definition 2.4.3 [67] Let C be subset of a Banach space X. A self-mapping $f : C \to C$ is called contraction on C of there exists a constant $\alpha \in (0, 1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contraction on C.

Theorem 2.4.4 [60] *Let X be a complete metric space and let T be a contraction of X into itself. Then T has a unique fixed point.*

In the following we will recall some useful operators and its properties.

Definition 2.4.5 [7] A mapping $T : C \to C$ is said to be

(1) *quasi–nonexpansive* with $F(T) \neq \emptyset$ if

$$\|Tx - p\| \leq \|x - p\| \quad \text{for all} \quad x \in C \quad \text{and} \quad p \in F(T).$$

(2) $\mathcal{L}$*–Lipschitz* if there exists a constant $\mathcal{L} > 0$ such that

$$\|Tx - Ty\| \leq \mathcal{L}\|x - y\| \quad \text{for all} \quad x, y \in C.$$

Definition 2.4.6 [59] A subset $\mathcal{K}$ of Banach space $\mathcal{E}$ is said to be retract of $\mathcal{E}$ if there is a continuous mapping $\mathcal{Q}$ from $\mathcal{E}$ onto $\mathcal{K}$ such that $\mathcal{Q}u = u$ for all $u \in \mathcal{K}$.

We call such $\mathcal{Q}$ a retraction of $\mathcal{E}$ onto $\mathcal{K}$. It follows that, if a mapping $\mathcal{Q}$ is a retraction, then $\mathcal{Q}v = v$ for all v in the range of $\mathcal{Q}$.

Definition 2.4.7 [59] A retraction $\mathcal{Q}$ is called a sunny if $\mathcal{Q}(\mathcal{Q}u+\lambda(u-\mathcal{Q}u)) = \mathcal{Q}u$ for all $u \in \mathcal{E}$ and $\lambda \geq 0$.

If a sunny retraction $\mathcal{Q}$ is also nonexpansive, then $\mathcal{K}$ is called a sunny nonexpansive retract of $\mathcal{E}$ [21].

Banach spaces are comparable to Hilbert spaces in that sunny nonexpansive retractions perform the same role. We note that if $\mathcal{K}$ is a closed and convex subset of a uniformly smooth Banach space $\mathcal{E}$ (the norm of $\mathcal{E}$ is Gateaux differentiable), then there exists and is unique a sunny nonexpansive retraction from $\mathcal{E}$ to $\mathcal{K}$.

Definition 2.4.8 [38] We call the space $\mathcal{E}$ satisfies the Opial's condition if for any sequence $\{x_n\}$ in $\mathcal{E}$, $x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$$

for all $y \in \mathcal{E}$ with $y \neq x$.

Definition 2.4.9 [7] A mapping $T : C \to \mathcal{E}$ is demiclosed at $y \in \mathcal{E}$ if for each sequence $\{x_n\}$ in C and each $x \in \mathcal{E}$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.

Lemma 2.4.10 [22] *Let C be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{E}$, and T a nonexpansive mapping on C. Then, $I - T$ is demiclosed at zero.*

Lemma 2.4.11 [56] *Suppose that $\mathcal{E}$ is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of $\mathcal{E}$ such that $\limsup_{n\to\infty} \|x_n\| \leq r$, $\limsup_{n\to\infty} \|y_n\| \leq r$ and $\limsup_{n\to\infty} \|t_n x_n + (1-t_n)y_n\| = r$ hold for some $r \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 2.4.12 [2] *Let $\mathcal{E}$ be a reflexive Banach space satisfying the Opial's condition, C a nonempty convex subset of $\mathcal{E}$, and $T : C \to X$ an operator such that $I - T$ demiclosed at zero and $F(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence in C such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ and $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

For a set–valued operator $\mathcal{A} : \mathcal{E} \to 2^{\mathcal{E}}$, we denote its domain, range and graph as follows:

$$\mathcal{D}(\mathcal{A}) = \{u \in \mathcal{E} : \mathcal{A}u \neq \emptyset\}, \mathcal{R}(\mathcal{A}) = \cup\{\mathcal{A}p : p \in \mathcal{D}(\mathcal{A})\}$$

and

$$\mathcal{G}(\mathcal{A}) = \{(u, v) \in \mathcal{E} \times \mathcal{E} : u \in \mathcal{D}(\mathcal{A}), v \in \mathcal{A}u\},$$

respectively. The inverse $\mathcal{A}^{-1}$ of $\mathcal{A}$ is defined by $u \in \mathcal{A}^{-1}v$, if and only if $v \in \mathcal{A}u$.

Definition 2.4.13 [60] $\mathcal{A}$ is called accretive if $\forall u_i \in \mathcal{D}(\mathcal{A})$ and $v_i \in \mathcal{A}u_i$ $(i = 1, 2)$, there exists $j = \mathcal{J}(u_1 - u_2)$ such that $\langle v_1 - v_2, j\rangle \geq 0$.

An accretive operator $\mathcal{A}$ in a Banach space $\mathcal{E}$ is said to satisfy the range condition if $\overline{\mathcal{D}(\mathcal{A})} \subset \mathcal{R}(I + \mu\mathcal{A})$ for all $\mu > 0$, where $\overline{\mathcal{D}(\mathcal{A})}$ denotes the closure of the domain of $\mathcal{A}$.

It is well known that for an accretive operator $\mathcal{A}$ which satisfies the range condition, $\mathcal{A}^{-1}0 = Fix(\mathcal{J}_\mu^{\mathcal{A}})$ for all $\mu > 0$.

Let $\mathcal{H}$ be a real Hilbert space. If $\mathcal{A} : \mathcal{E} \to 2^{\mathcal{E}}$ is an m-accretive operator[11, 12, 15], then $\mathcal{A}$ is called maximal accretive operator [18], and for all $\mu > 0$, $\mathcal{R}(\mathcal{I} + \mu\mathcal{A}) = \mathcal{H}$ if and only if $\mathcal{A}$ is called maximal monotone.[59] Denote by dom(h) the domain of a function $h : \mathcal{H} \to (-\infty, \infty]$, i.e.,

$$dom(h) = \{u \in \mathcal{H} : h(u) < \infty\}.$$

Lemma 2.4.14 [52] *Let $h \in \mathcal{T}_0(\mathcal{H})$. Then, ∂h is maximal monotone.*

Lemma 2.4.15 [66] *Let $\mathcal{E}$ be a Banach space, and $p > 1$and $R > 0$ be two fixed numbers. Then, $\mathcal{E}$ is uniformly convex if and only if there exists a continuous, strictly increasing, and convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that*

$$\|\alpha u + (1-\alpha)v\|^p \leq \|u\|^p + (1-\alpha)\|v\|^p - \alpha(1-\alpha)\varphi(\|u - v\|),$$

for all $u, v \in \mathcal{B}_\lambda[0]$ and $\alpha \in [0, 1]$.

Lemma 2.4.16 [54] *Let $\mathcal{K}$ be a nonempty subset of a Banach space $\mathcal{E}$ and $\mathcal{S} : \mathcal{K} \to \mathcal{E}$ a uniformly continuous mapping. Let $\{u_n\} \subset \mathcal{K}$ be an approximating fixed point sequence of $\mathcal{S}$, i.e. $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$. Then, $\{v_n\}$ is an approximating fixed point sequence of $\mathcal{S}$ whenever $\{v_n\}$ is in $\mathcal{K}$ such that $\lim_{n\to\infty} \|u_n - v_n\| = 0$.*

Lemma 2.4.17 [12] *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{E}$. If $\mathcal{S} : \mathcal{K} \to \mathcal{E}$ is a nonexpansive mapping, then $\mathcal{I} - \mathcal{S}$ has the demiclosed property with respect to 0.*

2.5 Rate of convergence

Definition 2.5.1 [49] Let C be a closed interval on the real line and let $f : C \to C$ be a continuous mapping. Suppose that $\{x_n\}$ and $\{w_n\}$ are two iterations which converge to the fixed point p of f. Then $\{x_n\}$ is said to converge faster than $\{w_n\}$ if

$$|x_n - p| \leq |w_n - p|$$

for all $n \geq 1$.

The following definitions about the rate of convergence are due to Berinde [8].

Definition 2.5.2 [8] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers converging to a and b respectively. If $\lim_{n\to\infty} \frac{|a_n-a|}{|b_n-b|} = 0$, then $\{a_n\}$ converges faster than $\{b_n\}$.

Definition 2.5.3 [8] Suppose that for two fixed-point iteration processes $\{x_n\}$ and $\{u_n\}$, both converging to the same fixed point p, the error estimates

$$\|x_n - p\| \leq a_n \qquad \text{for all} \quad n \geq 1,$$
$$\|u_n - p\| \leq b_n \qquad \text{for all} \quad n \geq 1$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers converging to zero. If $\{a_n\}$ converges faster than $\{b_n\}$, then $\{x_n\}$ *converges faster than* $\{u_n\}$ to p.

2.6 Proximal operator

Definition 2.6.1 [33] Let function $f : X \to (-\infty, \infty]$. Then f is said to be **proper** if there exists $x \in X$ with $f(x) < \infty$.

Definition 2.6.2 [7] Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a closed proper convex function, The proximal operator $\text{prox}_f : \mathbb{R}^n \to \mathbb{R}^n$ of f is defined by

$$\text{prox}_f(v) = \arg\min_x (f(x) + (1/2)\|x - v\|_2^2),$$

and the proximal operator of the scalar function αf, where $\alpha > 0$, which can be expressed as

$$\text{prox}_{\alpha f}(v) = \arg\min_x (f(x) + (1/2\alpha)\|x - v\|_2^2),$$

then $\text{prox}_{\alpha f}$ is call the proximal operator of f with parameter α.

Example 2.6.3 Let $f(x) = \frac{\gamma}{2}\|Ax - b\|^2$ when $A \in \mathbb{R}^{M\times N}$ and $\gamma > 0$ then

$$Prox_f(x) = (I + \gamma A^T A)^{-1}(x + \gamma A^T b).$$

Example 2.6.4 Let $f(x) = \|x\|_1$ when $x = (x_1, x_2, ..., x_N) \in \mathbb{R}^N$ then

$$Prox_f(x) = (\text{sign}(x_i)\text{max}|x_i| - 1, 0)_{i=1}^N.$$

Lemma 2.6.5 [19] *Let $r > 0$ and $R_{rf} = 2Prox_{rf} - I$ then R_{rf} is nonexpansive mapping.*

The following are a set of methods intended for regression in which the target value is expected to be a linear combination of the features. In mathematical notation, if $\hat{y}$ is the predicted value.

$$\hat{y}(w, x) = w_0 + w_1x_1 + ... + w_px_p.$$

Across the module, we designate the vector $w = (w_1, ..., w_p)$ as coef and w_0 as intercept to perform classification with generalized linear models.

Linear regression fits a linear model with coefficients $w = (w_1, ..., w_p)$ to minimize the residual sum of squares between the observed targets in the dataset, and the targets predicted by the linear approximation. Mathematically it solves a problem of the form:

$$\min_x \|Xw - y\|_2^2.$$

Linear regression will take in its fit method arrays X, y and will store the coefficients of the linear model in its coef member. The coefficient estimates for Ordinary Least Squares rely on the independence of the features. When features are correlated and the columns of the design matrix X have an approximately linear dependence, the design matrix becomes close to singular and as a result, the least-squares estimate becomes highly sensitive to random errors in the observed target, producing a large variance. This situation of multicollinearity can arise, for example, when data are collected without an experimental design.

Ridge regression addresses some of the problems of Ordinary Least Squares by imposing a penalty on the size of the coefficients. The ridge coefficients minimize a penalized residual sum of squares:

$$\min_{x} \|Xw - y\|_2^2 + \alpha \|w\|_2^2$$

the complexity parameter $\alpha \geq 0$ controls the amount of shrinkage: the larger the value of , the greater the amount of shrinkage and thus the coefficients become more robust to collinearity. As with other linear models, Ridge will take in its fit method arrays X, y and will store the coefficients of the linear model in its coef member.

The Lasso is a linear model that estimates sparse coefficients. It is useful in some contexts due to its tendency to prefer solutions with fewer non-zero coefficients, effectively reducing the number of features upon which the given solution is dependent. For this reason, Lasso and its variants are fundamental to the field of compressed sensing. Under certain conditions, it can recover the exact set of non-zero coefficients. Mathematically, it consists of a linear model with an added regularization term. The objective function to minimize is:

$$\min_{x} \|Xw - y\|_2^2 + \alpha \|w\|_1.$$

The Lasso estimate thus solves the minimization of the least-squares penalty with $\alpha \|w\|_1$ added, where α is a constant and $\|w\|_1$ is the $l_1 - norm$ of the coefficient vector.

CHAPTER III

MAIN RESULTS

3.1 Novel Noor iterations technique for solving nonlinear equations

3.1.1 Convergence theorem

In this section, we provide the convergence theorem of CT–iteration (1.0.6) for continuous functions on an arbitrary closed interval.

Now, we will give some crucial lemmas for proofs of our main results.

Lemma 3.1.1 *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (1.0.6). If $x_n \to a$, then a is a fixed point of f.*

Proof. Let $x_n \to a$, and suppose $a \neq f(a)$. Then $\{x_n\}$ is bounded. So, $\{f(x_n)\}$ is bounded by the continuity of f. So are $\{y_n\}, \{z_n\}, \{f(y_n)\}$ and $\{f(z_n)\}$. Moreover, $z_n \to a$ since $x_n \to a$ and $\mu_n \to 0$. We also have $y_n \to a$ since $x_n \to a$, $\beta_n \to 0$ and $\tau_n \to 0$. From (1.0.6), we get

$$\begin{aligned} x_{n+1} &= (1 - \gamma_n - \alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n) \\ &= z_n + \gamma_n(f(z_n) - z_n) + \alpha_n(f(y_n) - z_n). \end{aligned} \tag{3.1.1}$$

Let $p_k = f(z_k) - z_k, q_k = f(y_k) - z_k$. Then, we have

$$\begin{aligned} \lim_{k\to\infty} p_k &= \lim_{k\to\infty} (f(z_k) - z_k) = f(a) - a \neq 0, \\ \lim_{k\to\infty} q_k &= \lim_{k\to\infty} (f(y_k) - z_k) = f(a) - a \neq 0. \end{aligned}$$

From (3.1.1) we get

$$\begin{aligned} x_n &= z_1 + \sum_{k=1}^{n} \gamma_k (f(z_k) - z_k) + \sum_{k=1}^{n} \alpha_k (f(y_k) - z_k) \\ &= z_1 + \sum_{k=1}^{n} \gamma_k p_k + \sum_{k=1}^{n} \alpha_k q_k. \end{aligned}$$

It is worth noting here that $\sum_{k=1}^{\infty} \gamma_k p_k < \infty$ since $\lim_{k\to\infty} p_k \neq 0$ and $\sum_{k=1}^{\infty} \gamma_k < \infty$. This shows that $\{x_n\}$ is a divergent sequence since $\lim_{k\to\infty} q_k \neq 0$ and $\sum_{k=1}^{\infty} \alpha_k = \infty$. This contradicts to the convergence of $\{x_n\}$. Hence $f(a) = a$ and a is fixed point of f. □

Lemma 3.1.2 *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (1.0.6). If $\{x_n\}$ is bounded, then $\{x_n\}$ is convergent.*

Proof. Suppose $\{x_n\}$ is not convergent. Let $a = \liminf_n x_n$ and $b = \limsup_n x_n$. Then $a < b$. We first show that if $a < m < b$, then $f(m) = m$. Suppose $f(m) \neq m$. Without loss of generality, we suppose $f(m) - m > 0$. Since f is continuous, there exists δ with $0 < \delta < b - a$ such that for $|x - m| \leq \delta$, $f(x) - x > 0$. By continuity of f and $\{x_n\}$ is bounded we have that $\{f(x_n)\}$ is bounded, so $\{z_n\}, \{y_n\}$, $\{f(z_n)\}$ and $\{f(y_n)\}$ are bounded sequences. Using

$$\begin{aligned} x_{n+1} - x_n &= (1 - \gamma_n - \alpha_n)(z_n - x_n) + \gamma_n (f(z_n) - x_n) + \alpha_n (f(y_n) - x_n), \\ y_n - x_n &= \tau_n (f(x_n) - x_n) + \beta_n (f(z_n) - x_n), \\ z_n - x_n &= \mu_n (f(x_n) - x_n), \end{aligned}$$

we can easily show that $|z_n - x_n| \to 0$, $|y_n - x_n| \to 0$ and $|x_{n+1} - x_n| \to 0$. Thus, there exists a positive integer N such that

$$|x_{n+1} - x_n| < \frac{\delta}{2},\ |y_n - x_n| < \frac{\delta}{2},\ |z_n - x_n| < \frac{\delta}{2}, \forall n > N. \tag{3.1.2}$$

Since $b = \limsup_n x_n > m$, there exists $k_1 > N$ such that $x_{n_{k_1}} > m$. Let $n_{k_1} = k$, then $x_k > m$. For x_k, there exist two cases as follows:

(i) $x_k > m + \frac{\delta}{2}$, then $x_{k+1} > x_k - \frac{\delta}{2} \geq m$ using (3.1.2). So, we have $x_{k+1} > m$.

(ii) $m < x_k < m + \frac{\delta}{2}$, then $m - \frac{\delta}{2} < y_k < m + \delta$ and $m - \frac{\delta}{2} < z_k < m + \delta$ by (3.1.2). So, we obtain $|x_k - m| < \frac{\delta}{2} < \delta, |y_k - m| < \delta, |z_k - m| < \delta$. Hence

$$f(x_k) - x_k > 0,\ f(y_k) - y_k > 0,\ f(z_k) - z_k > 0. \tag{3.1.3}$$

In addition,

$$\begin{aligned} y_k - z_k &= (1 - \tau_k - \beta_k)(x_k - z_k) + \tau_k(f(x_k) - z_k) \\ &\quad + \beta_k(f(z_k) - z_k) \\ &= (1 - \tau_k - \beta_k)(x_k - z_k) + \tau_k(f(x_k) - x_k) \\ &\quad + \tau_k(x_k - z_k) + \beta_k(f(z_k) - z_k) \\ &= (1 - \beta_k)(x_k - z_k) + \tau_k(f(x_k) - x_k) \\ &\quad + \beta_k(f(z_k) - z_k) \\ &= (1 - \beta_k)\mu_k(x_k - f(x_k)) + \tau_k(f(x_k) - x_k) \\ &\quad + \beta_k(f(z_k) - z_k). \end{aligned} \tag{3.1.4}$$

From (3.1.1), (3.1.3) and (3.1.4), we have

$$x_{k+1} = z_k + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - z_k)$$

$$
\begin{aligned}
&= x_k + \mu_k(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) + \alpha_k(y_k - z_k) \\
&= x_k + \mu_k(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad + \alpha_k(-(1 - \beta_k)\mu_k(f(x_k) - x_k) + \tau_k(f(x_k) - x_k) + \beta_k(f(z_k) - z_k)) \\
&= x_k + \mu_k(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad - \alpha_k(1 - \beta_k)\mu_k(f(x_k) - x_k) + \alpha_k\tau_k(f(x_k) - x_k) + \alpha_k\beta_k(f(z_k) - z_k) \\
&= x_k + \mu_k(1 - \alpha_k(1 - \beta_k))(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad + \alpha_k\tau_k(f(x_k) - x_k) + \alpha_k\beta_k(f(z_k) - z_k) \\
&= x_k + \mu_k(1 - \alpha_k + \alpha_k\beta_k)(f(x_k) - x_k) + \gamma_k(f(z_k) - z_k) + \alpha_k(f(y_k) - y_k) \\
&\quad + \alpha_k\tau_k(f(x_k) - x_k) + \alpha_k\beta_k(f(z_k) - z_k) \\
&> x_k.
\end{aligned}
$$

Thus $x_{k+1} > x_k > m$. This together with (i) and (ii), imply $x_{k+1} > m$. Similarly, we get that $x_{k+2} > m$, $x_{k+3} > m, \ldots$. Thus we have $x_n > m$ for all $n > k = n_{k_1}$. So $a = \lim_{k\to\infty} x_{n_k} \geq m$, which is a contradiction with $a < m$. Thus $f(m) = m$.

We next consider the following two cases.

(i) There exists x_M such that $a < x_M < b$. Then $f(x_M) = x_M$. It follows that

$$z_M = (1 - \mu_M)x_M + \mu_M f(x_M) = x_M$$

and

$$
\begin{aligned}
y_M &= (1 - \tau_M - \beta_M)z_M + \tau_M f(x_M) + \beta_M f(z_M) \\
&= (1 - \tau_M - \beta_M)x_M + \tau_M f(x_M) + \beta_M f(x_M) \\
&= x_M.
\end{aligned}
$$

It follows that

$$x_{M+1} = (1 - \gamma_M - \alpha_M)z_M + \gamma_M f(z_M) + \alpha_M f(y_M)$$

$$= (1 - \tau_M - \gamma_M)x_M + \gamma_M f(x_M) + \alpha_M f(x_M)$$
$$= x_M.$$

Similarly, we obtain $x_M = x_{M+1} = x_{M+2} = \ldots$. It clear that $x_n \to x_M$. Since there exists $x_{n_k} \to a, x_M = a$. This shows that $x_n \to a$, which is a contradiction. (ii) For all $n, x_n \leq a$ or $x_n \geq b$. Since $b - a > 0$ and $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$, there exists $\tilde{N}$ such that $|x_{n+1} - x_n| < \frac{(b-a)}{2}$ for $n > \tilde{N}$. So, it is seen that $x_n \leq a$ for $n > \tilde{N}$, or it is always that $x_n \geq b$ for $n > \tilde{N}$. If $x_n \leq a$ for $n > \tilde{N}$, then $b = \lim_{j\to\infty} x_{n_j} \leq a$, which is a contradiction with $a < b$. If $x_n \geq b$ for $n > \tilde{N}$, then $a = \lim_{k\to\infty} x_{n_k} \geq b$, which is a contradiction with $a < b$. Thus we conclude that $x_n \to a$. The proof is completed. □

We are now ready to prove the main theorem.

Theorem 3.1.3 *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (1.0.6). Then $\{x_n\}$ is bounded if and only if it converges to a fixed point of f .*

Proof. Sufficiency is obvious. It suffices to show that if $\{x_n\}$ is bounded, then $\{x_n\}$ converges to a fixed point. Let $\{x_n\}$ be a bounded sequence. Using Lemma 3.1.2, we have $\{x_n\}$ is a convergent sequence. Hence, by Lemma 3.1.1, it converges to a fixed point of f. □

When $C = [a, b]$ in Theorem 3.1.3, we obtain the following result.

Corollary 3.1.4 *Let $f : [a, b] \to [a, b]$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$. Let $\{x_n\}$ be a sequence generated* *iterative by* $x_1 \in [a, b]$ *and*

$$z_n = (1-\mu_n)x_n + \mu_n f(x_n),$$

$$y_n = (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n),$$

$$x_{n+1} = (1-\gamma_n-\alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1,$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} \tau_n < \infty$. *Then* $\{x_n\}$ *converges to a fixed point of* f.

3.1.2 Rate of convergence

In this section, we compare the convergence rate of (1.0.6) with the CP–iteration proposed in [16]. We show that the CT–iteration (1.0.6) converges faster than the CP–iteration (1.0.5) for the class of continuous nondecreasing functions on an arbitrary interval in the sense of Rhoades [49].

We next prove some crucial lemmas which will be used in the sequel.

Lemma 3.1.5 *Let* C *be a closed interval on the real line and let* $f : C \to C$ *be a continuous and nondecreasing function. Let* $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ *and* $\{\tau_n\}$ *be sequences in* $[0, 1)$. *Let* $\{w_n\}$ *and* $\{x_n\}$ *be sequences defined by (1.0.5) and (1.0.6), respectively. Then the following hold:*

(i) If $f(w_1) < w_1$, *then* $f(w_n) < w_n$ *for all* $n \geq 1$ *and* $\{w_n\}$ *is nonincreasing.*

(ii) If $f(w_1) > w_1$, *then* $f(w_n) > w_n$ *for all* $n \geq 1$ *and* $\{w_n\}$ *is nondecreasing.*

(iii) If $f(x_1) < x_1$, *then* $f(x_n) < x_n$ *for all* $n \geq 1$ *and* $\{x_n\}$ *is nonincreasing.*

(iv) If $f(x_1) > x_1$, *then* $f(x_n) > x_n$ *for all* $n \geq 1$ *and* $\{x_n\}$ *is nondecreasing.*

Proof. (i) Let $f(w_1) < w_1$. Then $f(w_1) < r_1 \leq w_1$. Since f is nondecreasing, we have $f(r_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(r_1) < q_1 \leq w_1$. Thus $f(q_1) \leq f(w_1) < r_1 \leq w_1$. For q_1, we consider the following two cases.

Case 1: $f(r_1) < q_1 \leq r_1$. Then $f(q_1) \leq f(r_1) < q_1 \leq r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$, if $q_1 < w_2 \leq r_1$, then $f(w_2) \leq f(r_1) < q_1 < w_2$ and if $r_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < r_1 < w_2$. Thus we have $f(w_2) < w_2$.

Case 2: $r_1 < q_1 \leq w_1$. Then $f(q_1) \leq f(w_1) < r_1 \leq w_1$. This implies $f(q_1) < w_2 \leq w_1$. Thus $f(w_2) \leq f(w_1) < r_1 < q_1 \leq w_1$. It follows that if $f(q_1) < w_2 \leq q_1$, then $f(w_2) \leq f(q_1) < w_2$ and if $q_1 < w_2 \leq w_1$, then $f(w_2) \leq f(w_1) < q_1 < w_2$. Hence, we have $f(w_2) < w_2$.

In conclusion by Case 1 and Case 2, we have $f(w_2) < w_2$. By continuing in this way, we can show that $f(w_n) < w_n$ for all $n \geq 1$. This implies $r_n \leq w_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(r_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Thus $q_n \leq w_n$ for all $n \geq 1$, then $f(q_n) \leq f(w_n) < w_n$ for all $n \geq 1$. Hence, we have $w_{n+1} \leq w_n$ for all $n \geq 1$, that is $\{w_n\}$ is nonincreasing.

(ii) By using the same argument as in (i), we obtain the desired result.

(iii) Let $f(x_1) < x_1$. Then $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we have $f(z_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(z_1) < y_1 \leq x_1$. Thus $f(y_1) \leq f(x_1) < z_1 \leq x_1$. For y_1, we consider the following two cases.

Case 1: $f(z_1) < y_1 \leq z_1$. Then $f(y_1) \leq f(z_1) < z_1 < x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$, if $y_1 < x_2 \leq z_1$, then $f(x_2) \leq f(z_1) < y_1 < x_2$ and if $z_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < z_1 < x_2$. Thus we have $f(x_2) < x_2$.

Case 2: $z_1 < y_1 \leq x_1$. Then $f(y_1) \leq f(x_1) < z_1 \leq x_1$. This implies $f(y_1) < x_2 \leq x_1$. Thus $f(x_2) \leq f(x_1) < z_1 < y_1 \leq x_1$. It follows that if $f(y_1) < x_2 \leq y_1$, then $f(x_2) \leq f(y_1) < x_2$ and if $y_1 < x_2 \leq x_1$, then $f(x_2) \leq f(x_1) < y_1 < x_2$. Hence, we have $f(x_2) < x_2$.

In conclusion by Case 1 and Case 2, we have $f(x_2) < x_2$. By continuing in this way, we can show that $f(x_n) < x_n$ for all $n \geq 1$. This implies $z_n \leq x_n$ for all $n \geq 1$. Since f is nondecreasing, we have $f(z_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Thus $y_n \leq x_n$ for all $n \geq 1$, then $f(y_n) \leq f(x_n) < x_n$ for all $n \geq 1$. Hence, we have $x_{n+1} \leq x_n$ for all $n \geq 1$, that is $\{x_n\}$ is nonincreasing.

(iv) Following the proof line as in (iii), we obtain the desired result. □

Lemma 3.1.6 *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the CP–iteration (1.0.5) and CT–iteration (1.0.6), respectively. Then the following are satisfied:*
(i) If $f(w_1) < w_1$, then $x_n \leq w_n$ for all $n \geq 1$.
(ii) If $f(w_1) > w_1$, then $x_n \geq w_n$ for all $n \geq 1$.

Proof. (i) Let $f(w_1) < w_1$. Then $f(x_1) < x_1$ since $w_1 = x_1$. From (1.0.6), we get $f(x_1) < z_1 \leq x_1$. Since f is nondecreasing, we obtain $f(z_1) \leq f(x_1) < z_1 \leq x_1$. Hence $f(z_1) < y_1 \leq z_1$. Using the CP–iteration (1.0.5) and CT–iteration (1.0.6), we obtain the following estimation:

$$z_1 - r_1 = (1 - \mu_1)(x_1 - w_1) + \mu_1(f(x_1) - f(w_1)) = 0.$$

So, $z_1 = r_1$, and so

$$y_1 - q_1 = (1 - \tau_1 - \beta_1)(x_1 - w_1) + \tau_1(f(x_1) - r_1) + \beta_1(f(z_1) - f(r_1)) \leq 0.$$

Hence, we have $y_1 \leq q_1$. Since f is nondecreasing, we have $f(y_1) \leq f(q_1)$. We next obtain

$$x_2 - w_2 = (1 - \gamma_1 - \alpha_1)(z_1 - r_1) + \gamma_1(f(z_1) - q_1) + \alpha_1(f(y_1) - f(q_1)) \leq 0,$$

so, $x_2 \leq w_2$. Assume that $x_k \leq w_k$. Thus $f(x_k) \leq f(w_k)$. From Lemma 3.1.5

(i) and Lemma 3.1.5 (iii), we get $f(w_k) < w_k$ and $f(x_k) < x_k$. It follows that $f(x_k) < z_k \leq x_k$ and $f(z_k) \leq f(x_k) < z_k$. Thus

$$z_k - r_k = (1-\mu_k)(x_k - w_k) + \mu_k(f(x_k) - f(w_k)) \leq 0.$$

So, $z_k \leq r_k$. Since $f(z_k) \leq f(r_k)$, we have

$$y_k - q_k = (1-\tau_k-\beta_k)(x_k - w_k) + \tau_k(f(x_k) - r_k) + \beta_k(f(z_k) - f(r_k)) \leq 0,$$

so, $y_k \leq q_k$, which yields $f(y_k) \leq f(q_k)$. In addition, $f(z_k) \leq f(x_k) < z_k \leq x_k$, using (1.0.6), we have

$$f(z_k) - y_k = (1-\tau_k-\beta_k)(f(z_k) - x_k) + \tau_k(f(z_k) - f(x_k)) + \beta_k(f(z_k) - f(z_k)) \leq 0.$$

So, $f(z_k) - q_k = (f(z_k) - y_k) + (y_k - q_k) \leq 0.$

This shows that

$$x_{k+1} - w_{k+1} = (1-\gamma_k-\alpha_k)(z_k - r_k) + \gamma_k(f(z_k) - q_k) + \alpha_k(f(y_k) - f(q_k)) \leq 0,$$

which gives, $x_{k+1} \leq w_{k+1}$. By induction, we conclude that $x_n \leq w_n$ for all $n \geq 1$.

(ii) From Lemma 3.1.5 (ii), Lemma 3.1.5 (iv) and the same argument as in (i), we can show that $x_n \geq w_n$ for all $n \geq 1$. $\square$

For convenience, we write algorithm (1.0.6) by $CT(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$.

Proposition 3.1.7 *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 > \sup\{p \in C : p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. If $f(x_1) > x_1$, then $\{x_n\}$ defined by $CP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $CT(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3.1.5 $((ii), (iv))$, we know that $\{x_n\}$ is nondecreasing. Since the initial point $x_1 > \sup\{p \in C : p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Proposition 3.1.8 *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded with $x_1 < \inf\{p \in C : p = f(p)\}$. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. If $f(x_1) < x_1$, then $\{x_n\}$ defined by $CP(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ and $CT(x_1, \alpha_n, \beta_n, \mu_n, \gamma_n, \tau_n, f)$ do not converge to a fixed point of f.*

Proof. From Lemma 3.1.5 $((i), (iii))$, we know that $\{x_n\}$ is nonincreasing. Since the initial point $x_1 < \inf\{p \in C : p = f(p)\}$, it follows that $\{x_n\}$ does not converge to a fixed point of f. □

Next, we compare the rate of convergence of CT–iteration with CP–iteration.

Theorem 3.1.9 *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the CP–iteration (1.0.5) and the CT–iteration (1.0.6), respectively. If the CP–iteration $\{w_n\}$ converges to $p \in F(f)$, then the CT–iteration $\{x_n\}$ converges to p. Moreover, the CT–iteration (1.0.6) converges faster than the CP–iteration (1.0.5).*

Proof. Assume that the CP–iteration $\{w_n\}$ converges to $p \in F(f)$. Put $L = \inf\{p \in C : p = f(p)\}$ and $U = \sup\{p \in C : p = f(p)\}$. For $w_1 = x_1$, we devide our proof into the following three cases:

Case 1: $w_1 = x_1 > U$, Case 2: $w_1 = x_1 < L$, Case 3: $L \le w_1 = x_1 \le U$.

Case 1: $w_1 = x_1 > U$. By Proposition 3.1.7, we get $f(w_1) < w_1$ and $f(x_1) < x_1$. So, by Lemma 3.1.6 (i), we have $x_n \le w_n$ for all $n \ge 1$. By induction, we can show that $U \leq x_n$ for all $n \geq 1$. Then, we have $0 \leq x_n - p \leq w_n - p$, which yields $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. This shows that $x_n \to p$. By Definition 2.5.1, we conclude that the CT–iteration $\{x_n\}$ converges faster than the CP–iteration $\{w_n\}$.

Case 2: $w_1 = x_1 < L$. By Proposition 3.1.8, we get $f(w_1) > w_1$ and $f(x_1) > x_1$. This implies, by Lemma 3.1.6 (ii), that $x_n \geq w_n$ for all $n \geq 1$. So, by induction, we can show that $x_n \leq L$ for all $n \geq 1$. Then, we have $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. It follows that $x_n \to p$ and the CT–iteration $\{x_n\}$ converges faster than the CP–iteration $\{w_n\}$.

Case 3: $L \leq w_1 = x_1 \leq U$. Suppose that $f(w_1) \neq w_1$. If $f(w_1) < w_1$, we have, by Lemma 3.1.5 (i), that $\{w_n\}$ is nonincreasing with limit p. Lemma 3.1.6 (i) gives $p \leq x_n \leq w_n$ for all $n \geq 1$. It follows that $|x_n - p| \leq |w_n - p|$ for all $n \geq 1$. Therefore $x_n \to p$ and the result follows. If $f(w_1) > w_1$, by Lemma 3.1.5 (ii) and Lemma 3.1.6 (ii), then we can also show that the result holds. □

3.1.3 Numerical examples

In this section, some numerical examples are given to demonstrate the convergence of the algorithm defined in this paper. For convenience, we call the iteration (1.0.6) the CT–iteration.

Example 3.1.10 $f : [-1, 4] \to [-1, 4]$ defined by $f(x) = \frac{x^3+x-3}{19}$. The fixed point of the function is $p = -0.166925066$. Initial point is $x_1 = 4$ and control conditions are $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{1.7}}$, $\mu_n = \frac{1}{(n+1)^{2.3}}$, $\gamma_n = \frac{1}{(n+1)^{1.5}}$ and $\tau_n = \frac{1}{(n+1)^{1.1}}$. The stopping criteria is $|x_n - p| < 10^{-8}$.

Example 3.1.11 $f : [1, \infty] \to [1, \infty]$ defined by $f(x) = x^{0.3} - (\sqrt{log(x+9) - 1})^3$. The fixed point of the function is $p = 1$. Initial point is $x_1 = 9$ and control conditions are $\alpha_n = \frac{1}{(n+1)^{0.5}}$, $\beta_n = \frac{1}{(n+1)^{2.0}}$, $\mu_n = \frac{1}{(n+1)^{3.6}}$, $\gamma_n = \frac{1}{(n+1)^{2.5}}$ and $\tau_n = \frac{1}{(n+1)^{1.1}}$. The stopping criteria is $|x_n - p| < 10^{-6}$.

n	Mann	Ishikawa	Noor	CP	SP	CT-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\|x_n - p\|$
1	4	4	4	4	4	4	4.1669251
5	1.3932393	0.7533402	0.6286365	0.451696663	0.2932282	0.0012006	0.1681257
10	0.0461983	-0.0497313	-0.0662004	-0.090005411	-0.1191936	-0.1529447	0.0139804
15	-0.1207432	-0.1415242	-0.1450865	-0.150271594	-0.1573078	-0.1643412	0.0025839
20	-0.1538461	-0.1597323	-0.1607405	-0.162211234	-0.1643099	-0.1662565	0.0006686
25	-0.1625577	-0.1645237	-0.1648603	-0.165351661	-0.1660745	-0.1667147	0.0002104
30	-0.1652901	-0.1660262	-0.1661522	-0.166336214	0.1666125	-0.1668496	0.0000755
35	-0.1662585	-0.1665587	-0.1666100	-0.166685055	-0.1667994	-0.1668953	0.0000298
No. of iterations	133	126	124	119	113	97	

Table 3.1.1: Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = \frac{x^3+x-3}{19}$.

n	$\|x_n - p\|$							
	Initial points were close to p					Initial points were far from p		
	x_1= -0.3	x_1= -0.2	x_1= -0.1	x_1= 0	x_1= 0.1	x_1= 1	x_1= 2	x_1= 3
1	0.1330749	0.0330749	0.0669251	0.1669251	0.2669251	1.1669251	2.1669251	3.1669251
5	0.0001234	3.21×10^{-5}	6.72×10^{-5}	0.0001716	0.0002780	0.0008025	0.0023414	0.0209243
10	1.03×10^{-5}	2.71×10^{-6}	5.58×10^{-6}	1.43×10^{-5}	2.31×10^{-5}	6.70×10^{-5}	0.0001956	0.0017466
15	1.93×10^{-6}	5.29×10^{-7}	1.01×10^{-6}	2.61×10^{-6}	4.26×10^{-6}	1.23×10^{-5}	3.61×10^{-5}	0.0003229
20	5.26×10^{-7}	1.61×10^{-7}	2.35×10^{-7}	6.52×10^{-7}	1.07×10^{-6}	3.17×10^{-6}	9.38×10^{-6}	8.35×10^{-5}
25	1.88×10^{-7}	7.39×10^{-8}	5.10×10^{-8}	1.82×10^{-7}	3.16×10^{-7}	9.75×10^{-7}	2.97×10^{-6}	2.63×10^{-5}
30	8.91×10^{-8}	4.80×10^{-8}	3.21×10^{-9}	4.38×10^{-8}	9.18×10^{-8}	3.28×10^{-7}	1.08×10^{-6}	9.45×10^{-6}
35	5.54×10^{-8}	1.43×10^{-8}	2.15×10^{-8}	3.00×10^{-9}	1.59×10^{-8}	1.09×10^{-7}	4.49×10^{-7}	3.75×10^{-6}

Table 3.1.2: The sequences generated by CT–iteration for given x_1= -0.3, -0.2, -0.1, 0, 0.1, 1, 2 and 3 in Example 3.1.10.

Figure 3.1.1: Error values obtained from CT, Ishikawa, Noor, SP, CP and Mann iterations for given $x_1 = 4$ of $f(x) = \frac{x^3+x-3}{19}$.

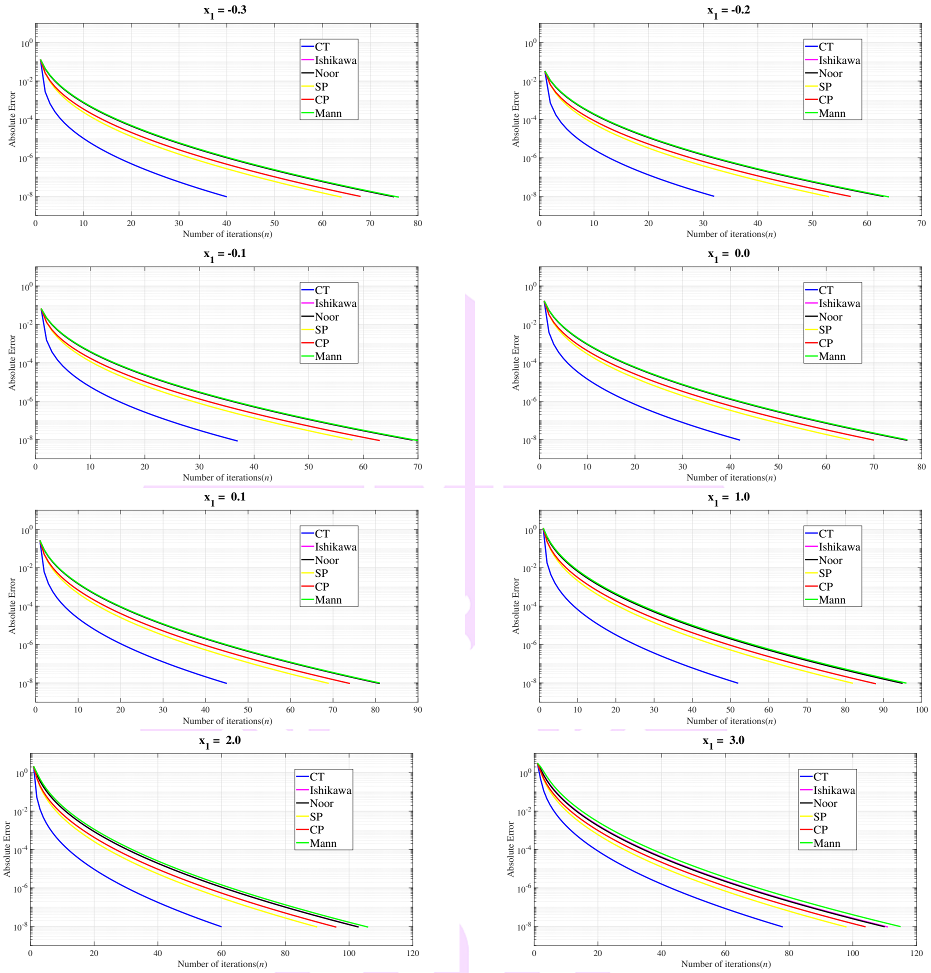

Figure 3.1.2: Convergence behaviors for given x_1= -0.3, -0.2, -0.1,0, 0.1, 1, 2 and 3 in Example 3.1.10.

n	Mann	Ishikawa	Noor	CP	SP	CT-iteration	
	u_n	s_n	l_n	w_n	h_n	x_n	$\|x_n - p\|$
1	9	9	9	9	9	9	8
5	1.5901350	1.5674767	1.5674116	1.4934240	1.3822427	1.2214054	0.2214054
10	1.1277881	1.1222912	1.1222778	1.1071548	1.0795988	1.0458060	0.0458060
15	1.0423403	1.0404612	1.0404568	1.0355209	1.0259652	1.0148665	0.0148665
20	1.0170313	1.0162644	1.0162626	1.0142875	1.0103511	1.0059037	0.0059037
25	1.0077061	1.0073563	1.0073555	1.0064638	1.0046562	1.0026478	0.0026478
30	1.0037823	1.0036097	1.0036093	1.0031722	1.002276	1.0012913	0.0012913
35	1.0019726	1.0018822	1.0018820	1.0016542	1.0011834	1.0006701	0.0006701
40	1.0010788	1.0010292	1.0010291	1.0009046	1.0006457	1.0003650	0.0003650
45	1.0006131	1.0005849	1.0005849	1.0005141	1.0003663	1.0002068	0.0002068
No. of iterations	124	123	123	121	116	108	

Table 3.1.3: Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$ and $x_1 = 9$.

n	$\|x_n - p\|$							
	Initial points were close to p					Initial points were far from p		
	x_1= 1.5	x_1= 2	x_1= 2.5	x_1= 3	x_1= 3.5	x_1= 15	x_1= 40	x_1= 70
1	0.5	1	1.5	2	2.5	14	39	69
5	0.0227557	0.0421025	0.0594148	0.0753317	0.0902155	0.3361598	0.7234206	1.1203709
10	0.0048476	0.008938	0.0125785	0.0159083	0.0190081	0.0685687	0.1417523	0.2124014
15	0.0015833	0.0029173	0.0041027	0.0051859	0.0061934	0.0221860	0.0454512	0.0675607
20	0.0006301	0.0011607	0.0016320	0.0020625	0.0024627	0.0088014	0.0179776	0.0266527
25	0.0002829	0.0005210	0.0007325	0.0009256	0.0011051	0.0039458	0.0080498	0.0119216
30	0.0001380	0.0002542	0.0003573	0.0004515	0.0005391	0.0016798	0.0039226	0.0058065
35	7.16×10^{-5}	0.0001319	0.0001854	0.0002343	0.0002798	0.0009983	0.0020349	0.0030114

Table 3.1.4: The sequences generated by CT-iteration for given x_1= 1.5, 2, 2.5, 3, 3.5, 15, 40 and 70 in Example 3.1.11.

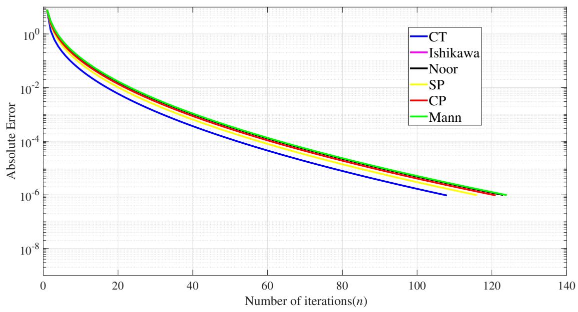

Figure 3.1.3: Mann, Ishikawa, Noor, SP, CP and CT iterations for given x_1= 9 of $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$.

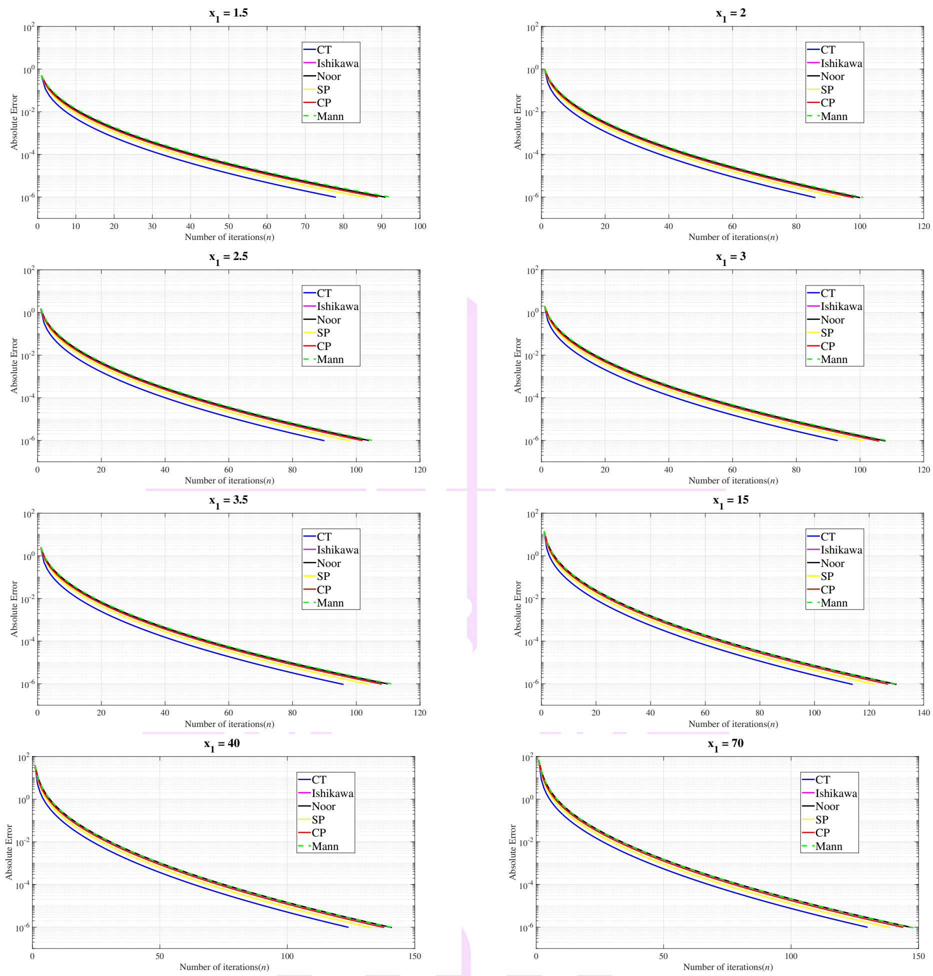

Figure 3.1.4: Convergence behaviors for given x_1= 1.5, 2, 2.5, 3, 3.5, 15, 40 and 70 in Example 3.1.11.

Table 3.1.2 and 3.1.4 confirm that the proposed method performs favorably with rapid convergence and Table 3.1.1, Table 3.1.3, Figure 3.1.1, 3.1.2, 3.1.3 and Figure 3.1.4 show the behavior of six comparative methods consisting of Mann iteration, Ishikawa iteration, Noor iteration, CP-iteration, SP-iteration and CT-iteration in converging to the fixed point of the numerical experiments. The results of the both examples indicates that the CT-iteration converges faster than the other methods. Even though the initial points are differently selected as shown on Figure 3.1.2 and Figure 3.1.4, the convergence of CT–iteration still be better than other methods. The effect of initial point being close to or far from p is not observed from these examples. The control sequnces $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ of the examples are chosen to satisfy on conditions with Corollary 3.1.4. The option of sequences is flexible for user application. However, the optimal chioce of them is an open problem to investigate.

Next, we will consider on the rate of convergence between the CT-iteration and the algorithm defined in this paper. The Definition 2.5.1 will be used to indicate the rate of covergence in the numrical aspects and results are scoped only on the Example 3.1.10 and Example 3.1.11.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 3.1.10 visually. as Figure 3.1.5

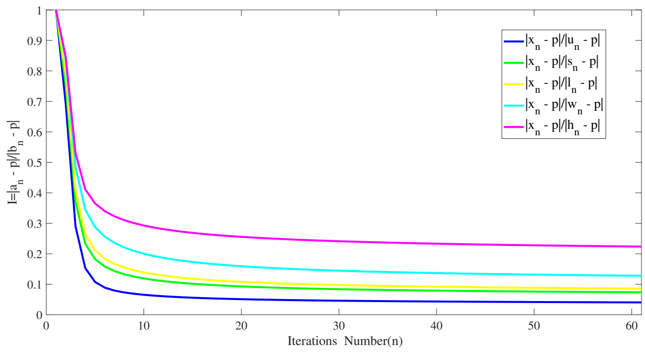

Figure 3.1.5: Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), CP–iteration (w_n) and SP-iteration $(h_n)$with CT–iteration (x_n) for Example 3.1.10.

We also give a graphic to compare the rates of convergence of the iterations mentioned in Example 3.1.11 visually. as Figure 3.1.6

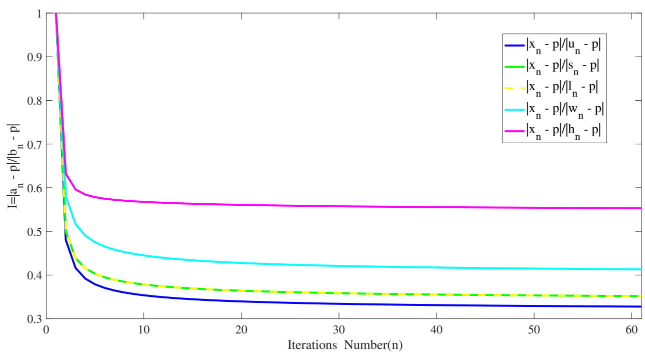

Figure 3.1.6: Convergence comparison of sequence generated by Mann iteration (u_n), Ishikawa iteration (s_n), Noor iteration (l_n), CP–iteration (w_n) and SP–iteration $(h_n)$with CT-iteration (x_n) for Example 3.1.11.

Table 3.1.5 and 3.1.7 show the absolute errors of Mann, Ishikawa, Noor, CP, SP and CT itertations of the Example 3.1.10 and Example 3.1.11, respectively. Table 3.1.6 and Table 3.1.8 show ratios between the absolute error of CT-iteration and those of other methods and graphs of Table 3.1.6 and Table 3.1.8 are represented on Figure 3.1.5 and Figure 3.1.6. Clearly, the graphs on both figures converge to constants less than 1. It indicates that the sequences of absolute error of CT-iteration are less than those sequences of other methods. By

Definition 2.5.1, we can conclude that CT-iteration converges to the fixed point faster than other method. These results verify the proof on the section 3.1.2 which show that CT-iteration converge faster than Mann, Ishikawa, Noor, CP, and SP itertations.

n	Mann $\lvert u_n - p\rvert$	Ishikawa $\lvert s_n - p\rvert$	Noor $\lvert l_n - p\rvert$	CP $\lvert w_n - p\rvert$	SP $\lvert h_n - p\rvert$	CT-iteration $\lvert x_n - p\rvert$
1	4.1669251	4.1669251	4.1669251	4.1669251	4.1669251	4.1669251
...	...	...	...	...	...	...
22	8.2979337E-03	4.5630635E-03	3.9235100E-03	2.6579550E-03	1.6397483E-03	4.1302723E-04
23	6.6660003E-03	3.6655020E-03	3.1517630E-03	2.1335990E-03	1.3103941E-03	3.2789365E-04
24	5.3826746E-03	2.9597038E-03	2.5448946E-03	1.7216513E-03	1.0529971E-03	2.6185811E-04
...	...	...	...	...	...	...
58	2.1520207E-05	1.1825202E-05	1.0168014E-05	6.8241504E-06	3.9132032E-06	8.7880715E-07
59	1.8878297E-05	1.0373410E-05	8.9196769E-06	5.9858026E-06	3.4293688E-06	8.7880712E-07
60	1.6580113E-05	9.1105165E-06	7.8337653E-06	5.2566342E-06	3.0089633E-06	6.7356212E-07

Table 3.1.5: The rate of convergence of Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = \frac{x^3+x-3}{19}$ given in Example 3.1.10.

Rate of convergence between two sequences

n	$\frac{\lvert x_n - p\rvert}{\lvert u_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert s_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert l_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert w_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert h_n - p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.1077615	0.1826927	0.2113296	0.2876608	0.3653690
10	0.0655970	0.1192918	0.1387968	0.2003645	0.2928943
20	0.0511132	0.0929416	0.1080929	0.1592907	0.2556291
30	0.0461208	0.0838958	0.0975696	0.1446444	0.2412792
40	0.0434849	0.0791196	0.0920146	0.1367792	0.2331997
50	0.0418128	0.0760882	0.0884893	0.1317378	0.2278379
60	0.0406258	0.0739361	0.0859869	0.1281362	0.2238863

Table 3.1.6: Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, CP–iteration and SP-iteration with CT–iteration (see in Table 3.1.5) for numerical experiment of Example 3.1.10.

n	Mann $\lvert u_n - p\rvert$	Ishikawa $\lvert s_n - p\rvert$	Noor $\lvert l_n - p\rvert$	CP $\lvert w_n - p\rvert$	SP $\lvert h_n - p\rvert$	CT-iteration $\lvert x_n - p\rvert$
1	8	8	8	8	8	8
...	...	...	...	...	...	...
35	1.9725555E-03	1.8822249E-03	1.8820245E-03	1.5884142E-03	1.1834264E-03	6.7014401E-04
36	1.7422239E-03	1.6623962E-03	1.6622192E-03	1.4029143E-03	1.0447074E-03	5.9139201E-04
37	1.5415695E-03	1.4708985E-03	1.4707419E-03	1.2413182E-03	9.2393975E-04	5.2285805E-04
...	...	...	...	...	...	...
86	1.4487895E-05	1.3817536E-05	1.3816066E-05	1.1662020E-05	8.5946173E-06	4.8221305E-06
87	1.3400554E-05	1.2780466E-05	1.2779106E-05	1.0786712E-05	7.9488484E-06	4.4593406E-06
88	1.2400552E-05	1.1826702E-05	1.1825443E-05	9.9817516E-06	7.3550156E-06	4.1257981E-06

Table 3.1.7: The rate of convergence of Mann, Ishikawa, Noor, CP, SP and CT iterations for $f(x) = x^{0.3} - (\sqrt{log(x+9)-1})^3$ given in Example 3.1.11.

Rate of convergence between two sequences

n	$\frac{\lvert x_n - p\rvert}{\lvert u_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert s_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert l_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert w_n - p\rvert}$	$\frac{\lvert x_n - p\rvert}{\lvert h_n - p\rvert}$
1	1.0000	1.0000	1.0000	1.0000	1.0000
5	0.3751776	0.3901578	0.3902026	0.4675022	0.5792274
10	0.3584529	0.3745653	0.3746062	0.4453414	0.5754611
20	0.3466405	0.3629859	0.3630247	0.4303450	0.5703508
40	0.3384200	0.3547035	0.3547413	0.4202962	0.5653881
60	0.3350960	0.3513080	0.3513454	0.4162480	0.5629381
80	0.3332460	0.3494062	0.3494434	0.4139888	0.5614158
100	0.3537939	0.3709719	0.3710114	0.4220645	0.5970553

Table 3.1.8: Convergence comparison of sequences generated by Mann iteration, Ishikawa iteration, Noor iteration, CP–iteration and SP-iteration with CT–iteration (see in Table 3.1.7) for numerical experiment of Example 3.1.11.

3.2 Numerical reckoning fixed points for nonexpansive mappings via a faster iteration process and its application to constrained minimization problems, split feasibility problems and image deblurring problems

3.2.1 Rate of Convergence

In this section, we show that the iteration process (1.0.12) converges faster than the iteration of Picard (1.0.7). To support our understanding using MATLAB software, we provide two numerical examples.

Theorem 3.2.1 *Let C be a nonempty closed convex subset of a norm space E. Let T be a contraction with a contraction factor $k \in (0,1)$ and fixed point p. Let $\{u_n\}$ be defined by the iteration process (1.0.7) and $\{x_n\}$ by (1.0.12), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\epsilon, 1-\epsilon]$ for all $n \in \mathbb{N}$ and for some ϵ in (0, 1). Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our process (1.0.12) converges faster than (1.0.7).*

Proof. Using (1.0.12), we have

$$\begin{aligned}
\|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n T x_n - p\| \\
&= \|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| \\
&\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\
&\leq (1-\gamma_n)\|x_n - p\| + k\gamma_n\|x_n - p\| \\
&= (1-\gamma_n + k\gamma_n)\|x_n - p\| \\
&= 1 - (\gamma_n - k\gamma_n)\|x_n - p\| \\
&= (1-(1-k)\gamma_n)\|x_n - p\|,
\end{aligned}$$

and so

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| \\
&= \|(1-\beta_n)(Tx_n - p) + \beta_n(Tz_n - p)\| \\
&\le (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\
&\le k(1-\beta_n)\|x_n - p\| + k\beta_n\|z_n - p\| \\
&= k(1-\beta_n)\|x_n - p\| + k\beta_n((1-(1-k)\gamma_n)\|x_n - p\|) \\
&= k(1-\beta_n)\|x_n - p\| + (k\beta_n - (1-k)k\beta_n\gamma_n)\|x_n - p\| \\
&= k((1-\beta_n) + \beta_n - (1-k)\beta_n\gamma_n)\|x_n - p\| \\
&= k(1-(1-k)\beta_n\gamma_n)\|x_n - p\|.
\end{aligned}$$

This implies that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\| \\
&= \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \\
&\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\| \\
&\le k(1-\alpha_n)\|x_n - p\| + k\alpha_n\|y_n - p\| \\
&= k(1-\alpha_n)\|x_n - p\| + k\alpha_n(k(1-(1-k)\beta_n\gamma_n)\|x_n - p\|) \\
&= k(1-\alpha_n)\|x_n - p\| + k^2\alpha_n(1-(1-k)\beta_n\gamma_n)\|x_n - p\| \\
&= k(1-\alpha_n) + k^2\alpha_n(1-(1-k))\beta_n\gamma_n\|x_n - p]\| \\
&= k((1-\alpha_n) + k\alpha_n(1-(1-k)\beta_n\gamma_n))\|x_n - p\| \\
&= k((1-\alpha_n) + k(\alpha_n - (1-k)\alpha_n\beta_n\gamma_n))\|x_n - p\| \\
&< k((1-\alpha_n) + (\alpha_n - (1-k)\alpha_n\beta_n\gamma_n))\|x_n - p\| \\
&= k(1-(1-k)\alpha_n\beta_n\gamma_n)\|x_n - p\|.
\end{aligned}$$

Repetition of above processes gives the following inequalities

$$\begin{cases} \|x_{n+1} - p\| & \leq k(1 - (1-k)\alpha_n\beta_n\gamma_n)\|x_n - p\|, \\ \|x_n - p\| & \leq k(1 - (1-k)\alpha_{n-1}\beta_{n-1}\gamma_{n-1})\|x_{n-1} - p\|, \\ \|x_{n-1} - p\| & \leq k(1 - (1-k)\alpha_{n-2}\beta_{n-2}\gamma_{n-2})\|x_{n-2} - p\|, \\ & \vdots \\ \|x_2 - p\| & \leq k(1 - (1-k)\alpha_1\beta_1\gamma_1)\|x_1 - p\|, \\ \|x_1 - p\| & \leq k(1 - (1-k)\alpha_0\beta_0\gamma_0)\|x_0 - p\|. \end{cases}$$

From above inequalities, we derive

$$\|x_{n+1} - p\| \leq \|x_0 - p\| k^{n+1} \prod_{j=0}^{n} (1 - (1-k)\alpha_j\beta_j\gamma_j).$$

It follows that

$$\|x_{n+1} - p\| \leq \|x_0 - p\| k^{n+1} (1 - (1-k)\alpha\beta\gamma)^{n+1}$$

for all $n \in \mathbb{N}$ and for some $\alpha, \beta, \gamma > 0$ such that $\alpha \leq \alpha_n < 1, \beta \leq \beta_n < 1$ and $\gamma \leq \gamma_n < 1$. By the definition of the Picard iteration process, we have

$$\|u_{n+1} - p\| \leq k\|u_n - p\|$$

for all $n \in \mathbb{N}$. Note that $\|u_{n+1} - p\| \leq \|u_0 - p\| k^{n+1}$. Let

$$a_n = k^{n+1}\|u_0 - p\|$$

and

$$b_n = k^{n+1}(1 - (1-k)\alpha\beta\gamma)^{n+1}\|x_0 - p\|.$$

Then

$$\begin{aligned}\frac{b_n}{a_n} &= \frac{k^{n+1}(1-(1-k)\alpha\beta\gamma)^{n+1}\|x_0-p\|}{k^{n+1}\|u_0-p\|} \\ &= \frac{(1-(1-k)\alpha\beta\gamma)^{n+1}\|x_0-p\|}{\|u_0-p\|}.\end{aligned}$$

Define $\theta_n = (1-(1-k)\alpha\beta\gamma)^{n+1}$. Therefore, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{\theta_{n+1}}{\theta_n} &= \frac{(1-(1-k)\alpha\beta\gamma)^{n+2}}{(1-(1-k)\alpha\beta\gamma)^{n+1}} \\ &= 1-(1-k)\alpha\beta\gamma \\ &< 1.\end{aligned}$$

It thus follows from well-known ratio test that $\sum_{n=0}^{\infty}\theta_n < \infty$. Hence, we have $\lim_{n\to\infty}\theta_n = 0$ which implies that

$$\lim_{n\to\infty}\frac{b_n}{a_n} = 0.$$

Consequently $\{x_n\}$ converges faster than $\{u_n\}$. □

Now, we present an example which shows that our iteration process (1.0.12) converges at a rate faster than Agarwal et al. iteration process (1.0.11), Mann iteration process (1.0.8), Ishikawa iteration process (1.0.9), Noor iteration process (1.0.10) and Picard iteration process (1.0.7).

Example 3.2.2 Let $E = \mathbb{R}$ and $C = [1, 50]$. Let $T : C \to C$ be a mapping, which is defined by $T(x) = \sqrt{x^2 - 8x + 40}$ for all $x \in C$. Choose $\alpha_n = 0.85, \beta_n = 0.65, \gamma_n = 0.45$, with the initial value $x_1 = 40$. The corresponding our iteration process, Agarwal et al. iteration process, Noor iteration process, Ishikawa iteration process, Mann iteration process and Picard iteration processes are respectively given in Table 3.2.1.

Step	Picard	Mann	Ishikawa	Noor	Agarwal	New iteration
1	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000	40.0000000000
2	36.3318042492	36.8820336118	34.8751575132	33.9816211055	34.3249281505	32.3527454021
3	32.7008496221	33.7905308732	29.8335259837	28.0887816012	28.7529148550	24.9217277329
4	29.1159538575	30.7306375124	24.9067432334	22.3811620460	23.3289757744	17.8627609012
5	25.5892777970	27.7090706072	20.1467307646	16.9736024952	18.1321892967	11.5463790857
6	22.1381326176	24.7347891266	15.6449263114	12.0962209155	13.3147454600	6.9395376930
7	18.7880774656	21.8200359935	11.5741197024	8.2289280979	9.1939307941	5.2225332988
8	15.5784221001	18.9820007784	8.2638548016	6.0182077910	6.3717274607	5.0138389524
9	12.5721859009	16.2455313784	6.1736938982	5.2517005165	5.2434387591	5.0007808271
10	9.8733161157	13.6475866165	5.3185408455	5.0576355955	5.0298139084	5.0007808271
11	7.6482574613	11.2442765494	5.0768890301	5.0129587850	5.0033662656	5.0000024535
12	6.1081734180	9.1201110370	5.0179832209	5.0029016212	5.0003761718	5.0000001375
13	5.3333287129	7.3913650188	5.0041744485	5.0006491038	5.0000419870	5.0000000077
14	5.0771808572	6.1732610225	5.0009673150	5.0001451769	5.0000046858	5.0000000004
15	5.0160062399	5.4814708358	5.0002240577	5.0000324684	5.0000005229	5.0000000000
16	5.0032258274	5.1725897008	5.0000518932	5.0000072614	5.0000000584	5.0000000000
17	5.0006461643	5.0576419946	5.0000120186	5.0000016240	5.0000000065	5.0000000000
18	5.0001292729	5.0187159301	5.0000027835	5.0000003632	5.0000000007	5.0000000000
19	5.0000258562	5.0060176595	5.0000006447	5.0000000812	5.0000000001	5.0000000000
20	5.0000051713	5.0019286052	5.0000001493	5.0000000182	5.0000000000	5.0000000000
21	5.0000010343	5.0006174572	5.0000000346	5.0000000041	5.0000000000	5.0000000000
22	5.0000002069	5.0001976174	5.0000000080	5.0000000009	5.0000000000	5.0000000000
23	5.0000000414	5.0000632408	5.0000000019	5.0000000002	5.0000000000	5.0000000000
24	5.0000000083	5.0000202374	5.0000000004	5.0000000000	5.0000000000	5.0000000000
25	5.0000000017	5.0000064760	5.0000000001	5.0000000000	5.0000000000	5.0000000000
26	5.0000000003	5.0000020723	5.0000000000	5.0000000000	5.0000000000	5.0000000000
27	5.0000000001	5.0000006631	5.0000000000	5.0000000000	5.0000000000	5.0000000000
28	5.0000000000	5.0000002122	5.0000000000	5.0000000000	5.0000000000	5.0000000000
29	5.0000000000	5.0000000679	5.0000000000	5.0000000000	5.0000000000	5.0000000000
30	5.0000000000	5.0000000217	5.0000000000	5.0000000000	5.0000000000	5.0000000000
31	5.0000000000	5.0000000070	5.0000000000	5.0000000000	5.0000000000	5.0000000000
32	5.0000000000	5.0000000022	5.0000000000	5.0000000000	5.0000000000	5.0000000000
33	5.0000000000	5.0000000007	5.0000000000	5.0000000000	5.0000000000	5.0000000000
34	5.0000000000	5.0000000002	5.0000000000	5.0000000000	5.0000000000	5.0000000000
35	5.0000000000	5.0000000001	5.0000000000	5.0000000000	5.0000000000	5.0000000000
36	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000	5.0000000000

Table 3.2.1: Comparative results.

All sequences converges to $x^* = 5$. Comparison shows that our iteration process (1.0.12) requires least number of iterations among all the iterations mentioned above.

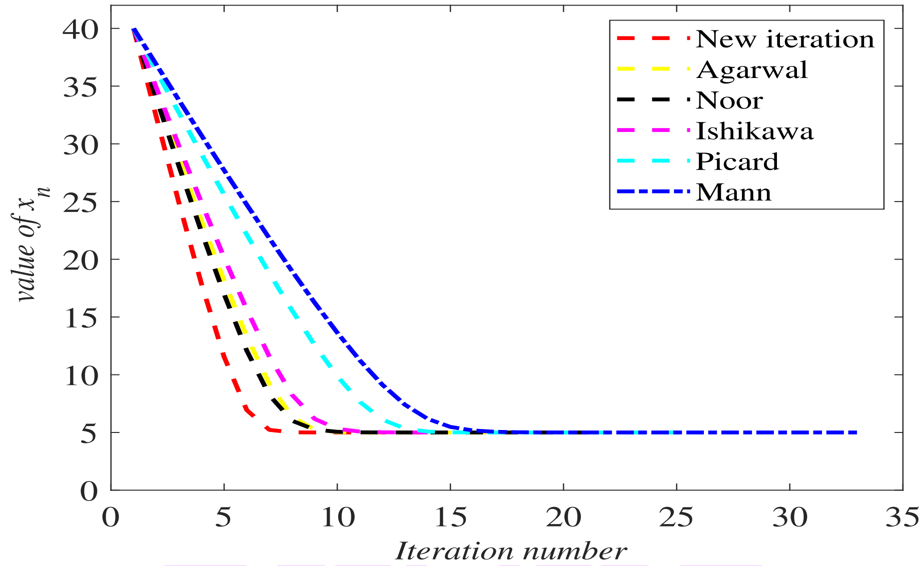

Figure 3.2.1: Convergence behavior of the Mann, the Picard, the Ishikawa, the Noor, the Argarwal and new iterations for the function given in Example 3.2.2

Example 3.2.3 Let $E = \mathbb{R}$ and $C = [1, 50]$. Let $T : C \to C$ be a mapping, which is defined by $T(x) = \sqrt{x^2 - 9x + 54}$ for all $x \in C$. Choose $\alpha_n = \beta_n = \gamma_n = \frac{3}{4}$, with the initial value $x_1 = 30$. The corresponding our iteration process, Agarwal et al. iteration process, Noor iteration process, Ishikawa iteration process, Mann iteration process and Picard iteration processes are respectively given in Table 3.2.2.

Step	Picard	Mann	Ishikawa	Noor	Agarwal	New iteration
1	30.0000000000	30.0000000000	30.0000000000	30.0000000000	30.0000000000	30.0000000000
2	26.1533936612	27.1150452459	25.0119824036	23.4891033190	24.0503308189	31.8347714371
3	22.4191761010	24.2907437151	20.2547559071	17.4668190633	18.4372719353	14.5055413771
4	18.8373796516	21.5420343135	15.8509087868	12.3265857284	13.3938203603	8.9907882699
5	15.4696624163	18.8892775011	12.0133051549	8.7275766163	9.3725555853	6.5301998399
6	12.4130372403	16.3606498049	9.0688620373	6.9585711603	6.9939357160	6.0555659291
7	9.8166266286	13.9954171304	7.2820400289	6.3102146269	6.1862068754	6.0050669583
8	7.8750567432	11.8475686983	6.4668031480	6.0979255677	6.0283693653	6.0045473127
9	6.7187058292	9.9869851099	6.1600652383	6.0306808428	6.0041338820	6.0000407497
10	6.2187342406	8.4900396666	6.0537250393	6.0095903071	6.0095981884	6.0000407497
11	6.0583865336	7.4083030742	6.0179028366	6.0029956076	6.0000864719	6.0000032715
12	6.0148623083	6.7246651786	6.0059514305	6.0009354914	6.0000124982	6.0000000293
13	6.0037328233	6.3468134658	6.0019768478	6.0002921200	6.0000018064	6.0000000026
14	6.0009342942	6.1586728531	6.0006564620	6.0000912177	6.0000002611	6.0000000002
15	6.0002336418	6.0708846663	6.0002179755	6.0000284834	6.0000000377	6.0000000000
16	6.0000584147	6.0313055772	6.0000723757	6.0000088941	6.0000000055	6.0000000000
17	6.0000146039	6.0137535390	6.0000240311	6.0000027772	6.0000000008	6.0000000000
18	6.0000036510	6.0060282506	6.0000079791	6.0000008672	6.0000000001	6.0000000000
19	6.0000009128	6.0026394884	6.0000026493	6.0000002708	6.0000000000	6.0000000000
20	6.0000002282	6.0011551843	6.0000008797	6.0000000846	6.0000000000	6.0000000000
21	6.0000000570	6.0005054713	6.0000002921	6.0000000264	6.0000000000	6.0000000000
22	6.0000000143	6.0002211587	6.0000000970	6.0000000082	6.0000000000	6.0000000000
23	6.0000000036	6.0000967598	6.0000000322	6.0000000026	6.0000000000	6.0000000000
24	6.0000000009	6.0000423330	6.0000000107	6.0000000008	6.0000000000	6.0000000000
25	6.0000000002	6.0000185208	6.0000000035	6.0000000003	6.0000000000	6.0000000000
26	6.0000000001	6.0000081029	6.0000000012	6.0000000001	6.0000000000	6.0000000000
27	6.0000000000	6.0000025450	6.0000000004	6.0000000000	6.0000000000	6.0000000000
28	6.0000000000	6.0000015509	6.0000000001	6.0000000000	6.0000000000	6.0000000000
29	6.0000000000	6.0000006785	6.0000000000	6.0000000000	6.0000000000	6.0000000000
30	6.0000000000	6.0000002969	6.0000000000	6.0000000000	6.0000000000	6.0000000000
31	6.0000000000	6.0000001299	6.0000000000	6.0000000000	6.0000000000	6.0000000000
32	6.0000000000	6.0000000586	6.0000000000	6.0000000000	6.0000000000	6.0000000000
33	6.0000000000	6.0000000249	6.0000000000	6.0000000000	6.0000000000	6.0000000000
34	6.0000000000	6.0000000109	6.0000000000	6.0000000000	6.0000000000	6.0000000000
35	6.0000000000	6.0000000048	6.0000000000	6.0000000000	6.0000000000	6.0000000000
36	6.0000000000	6.0000000021	6.0000000000	6.0000000000	6.0000000000	6.0000000000
37	6.0000000000	6.0000000009	6.0000000000	6.0000000000	6.0000000000	6.0000000000
38	6.0000000000	6.0000000004	6.0000000000	6.0000000000	6.0000000000	6.0000000000
39	6.0000000000	6.0000000002	6.0000000000	6.0000000000	6.0000000000	6.0000000000
40	6.0000000000	6.0000000001	6.0000000000	6.0000000000	6.0000000000	6.0000000000
41	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000	6.0000000000

Table 3.2.2: Comparative results.

All sequences converges to $x^* = 6$. Comparison shows that our iteration process (1.0.12) requires least number of iterations among all the iterations mentioned above.

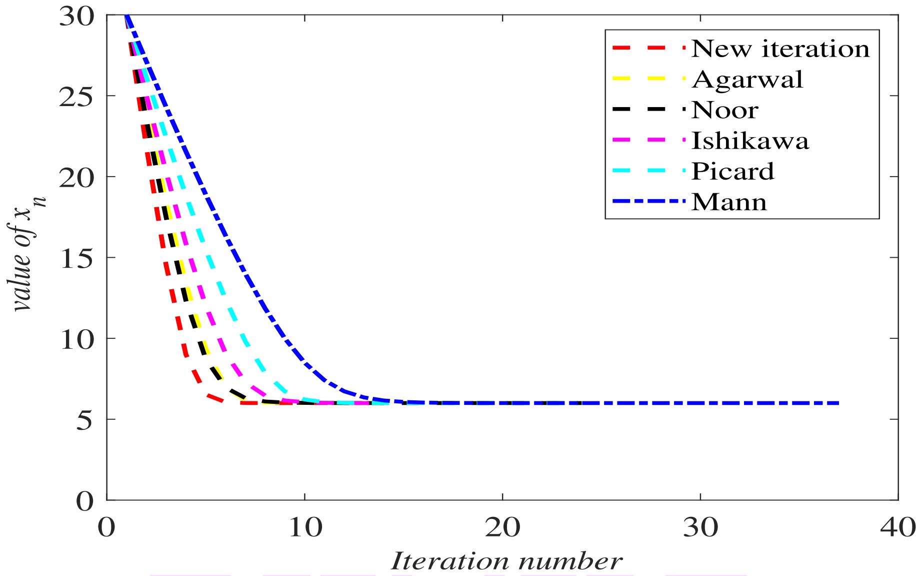

Figure 3.2.2: Convergence behavior of the Mann, the Picard, the Ishikawa, the Noor, the Argarwal and new iterations for the function given in Example 3.3.

3.2.2 Convergence theorems

In this section, we give some convergence theorems using our iteration process (1.0.12); please, see Table 3.2.1, Table 3.2.2, Figure 3.2.1 and Figure 3.2.1. Before proving the main theorems, we have the following lemmas.

Lemma 3.2.4 *Let C be a nonempty closed convex subset of a normed linear space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence defined by (1.0.12) and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Proof. Let $p \in F(T)$. From (1.0.12), we have

$$\begin{aligned} \|z_n - p\| &= \|(1-\gamma_n)x_n + \gamma_n Tx_n - p\| \\ &= \|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|Tx_n - p\| \\ &\leq (1-\gamma_n)\|x_n - p\| + \gamma_n\|x_n - p\| \\ &= \|x_n - p\| \end{aligned} \tag{3.2.1}$$

and

$$\begin{aligned}
\|y_n - p\| &= \|(1-\beta_n)Tx_n + \beta_n Tz_n - p\| \\
&= \|(1-\beta_n)(Tx_n - p) + \beta_n(Tz_n - p)\| \\
&\leq (1-\beta_n)\|Tx_n - p\| + \beta_n\|Tz_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|z_n - p\| \\
&\leq (1-\beta_n)\|x_n - p\| + \beta_n\|x_n - p\| \\
&= \|x_n - p\|. \qquad (3.2.2)
\end{aligned}$$

By using (3.2.1) and (3.2.2), we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\| \\
&= \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \\
&\leq (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|Ty_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|y_n - p\| \\
&\leq (1-\alpha_n)\|x_n - p\| + \alpha_n\|x_n - p\| \\
&= \|x_n - p\|.
\end{aligned}$$

This implies that $\{\|x_n - p\|\}$ is bounded and non-increasing for all $p \in F(T)$. Hence $\lim_{n\to\infty} \|x_n - p\|$ exists, as required. □

Lemma 3.2.5 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence given by (1.0.12) and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Proof. By Lemma 3.2.4, we see that $\lim_{n\to\infty} \|x_n - p\|$ exists. Assume that $\lim_{n\to\infty} \|x_n - p\| = c$. Using (3.2.1) and (3.2.2), we have

$$\limsup_{n\to\infty} \|y_n - p\| \leq c \qquad (3.2.3)$$

and

$$\limsup_{n\to\infty} \|z_n - p\| \leq c. \tag{3.2.4}$$

Since T is nonexpensive mapping, we have

$$\|Tx_n - p\| \leq \|x_n - p\|, \ \|Ty_n - p\| \leq \|y_n - p\| \text{ and } \|Tz_n - p\| \leq \|z_n - p\|.$$

Taking lim sup on both sides, using (3.2.3) and (3.2.4), we obtain

$$\limsup_{n\to\infty} \|Tx_n - p\| \leq c,$$
$$\limsup_{n\to\infty} \|Ty_n - p\| \leq c$$

and

$$\limsup_{n\to\infty} \|Tz_n - p\| \leq c. \tag{3.2.5}$$

Since

$$\begin{aligned} c &= \lim_{n\to\infty} \|x_{n+1} - p\| \\ &= \lim_{n\to\infty} \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\|, \end{aligned}$$

by using (3.2.5) and Lemma (2.4), we have

$$\lim_{n\to\infty} \|Tx_n - Ty_n\| = 0. \tag{3.2.6}$$

Now

$$\begin{aligned} \|x_{n+1} - p\| &= \|(1-\alpha_n)Tx_n + \alpha_n Ty_n - p\| \\ &= \|(1-\alpha_n)(Tx_n - p) + \alpha_n(Ty_n - p)\| \\ &= \|(Tx_n - p) - \alpha_n(Tx_n - p) + \alpha_n(Ty_n - p)\| \\ &= \|(Tx_n - p) + \alpha_n((Ty_n - p) - (Tx_n - p))\| \end{aligned}$$

$$= \|(Tx_n - p) + \alpha_n(Ty_n - p - Tx_n + p)\|$$
$$= \|(Tx_n - p) + \alpha_n(Ty_n - Tx_n)\|$$
$$\leq \|Tx_n - p\| + \alpha_n\|Ty_n - Tx_n\|.$$

Using (3.2.6), we have

$$c \leq \liminf_{n\to\infty} \|Tx_n - p\|. \tag{3.2.7}$$

It follows from (3.2.5) and (3.2.7) that

$$\lim_{n\to\infty} \|Tx_n - p\| = c. \tag{3.2.8}$$

On the other hand, we have

$$\|Tx_n - p\| = \|Tx_n - Ty_n + Ty_n - p\|$$
$$\leq \|Tx_n - Ty_n\| + \|Ty_n - p\|$$
$$\leq \|Tx_n - Ty_n\| + \|y_n - p\|,$$

and this yields that

$$c \leq \liminf_{n\to\infty} \|y_n - p\|. \tag{3.2.9}$$

Form (3.2.3) and (3.2.9), we get

$$\lim_{n\to\infty} \|y_n - p\| = c.$$

Since

$$c = \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} \|(1-\beta_n)(Tx_n - p) + \beta_n(Tz_n - p)\|. \tag{3.2.10}$$

From (3.2.5) and (3.2.10), by using Lemma 2.4.11 we obtain

$$\lim_{n\to\infty} \|Tz_n - Tx_n\| = 0. \tag{3.2.11}$$

In addition,

$$\begin{aligned} \|Tx_n - p\| &\le \|Tx_n - Tz_n + Tz_n - p\| \\ &\le \|Tx_n - Tz_n\| + \|Tz_n - p\| \\ &\le \|Tx_n - Tz_n\| + \|z_n - p\|. \end{aligned} \tag{3.2.12}$$

Using (3.2.8), (3.2.11) and (3.2.12), we have

$$c \le \liminf_{n\to\infty} \|z_n - p\|. \tag{3.2.13}$$

By (3.2.4) and (3.2.13), we obtain $\lim_{n\to\infty} \|z_n - p\| = c$. Thus

$$\begin{aligned} c &= \lim_{n\to\infty} \|z_n - p\| \\ &= \lim_{n\to\infty} \|(1-\gamma_n)(x_n - p) + \gamma_n(Tx_n - p)\|, \end{aligned}$$

gives by Lemma 2.4.11 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

By using Lemma 2.4.10, Lemma 2.4.12, Lemma 3.2.4 and Lemma 3.2.5, we will establish the following theorems.

Theorem 3.2.6 *Let E be a real uniformly convex Banach space which satisfies the Opial's condition, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by iteration process (1.0.12). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $p \in F(T)$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. For, let u and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By $\lim_{n\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0$ and $I - T$ is demiclosed with respect to zero by Lemma 2.4.10, therefore we obtain $Tu = u$. Again in the same manner, we can prove that $v \in F(T)$. Next, we prove the uniqueness. From Lemma 3.2.4 the limits $\lim_{n\to\infty} \|x_n - v\|$ exists. For this suppose that $u \neq v$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty} \|x_n - u\| &= \lim_{n_i\to\infty} \|x_{n_i} - u\| < \lim_{n_i\to\infty} \|x_{n_i} - v\| = \lim_{n\to\infty} \|x_n - v\| \\ &= \lim_{n_j\to\infty} \|x_{n_j} - v\| < \lim_{n_j\to\infty} \|x_{n_j} - u\| = \lim_{n\to\infty} \|x_n - u\|.\end{aligned}$$

This is a contradiction, so $u = v$. Hence, $\{x_n\}$ converges weakly to a fixed point of $F(T)$ and this completes the proof. □

Theorem 3.2.7 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (1.0.12) and $F(T) \neq \emptyset$. Then $\{x_n\}$ converges to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.*

Proof. Necessity is obvious. Suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. As proved in Lemma 3.2.5, $\lim_{n\to\infty} \|x_n - w\|$ exists for all $w \in F(T)$, therefore $\lim_{n\to\infty} d(x_n, F(T))$ exists. But by hypothesis, we have $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, therefore we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. We will show that $\{x_n\}$ is a Cauchy sequence in C. Since $\lim_{n\to\infty} d(x_n, F(T)) = 0$, for given $\epsilon > 0$, there exists n_0 in $\mathbb{N}$ such that for all $n \geq n_0$,

$$d(x_n, F(T)) < \frac{\epsilon}{2}.$$

Particularly, $\inf\{\|x_{n_0} - p\| : p \in F(T)\} < \frac{\epsilon}{2}$. Hence, there exist $p^* \in F(T)$ such that $\|x_{n_0} - p^*\| < \frac{\epsilon}{2}$. Now, for $m, n \geq n_0$,

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \leq 2\|x_{n_0} - p^*\| < \epsilon.$$

Hence $\{x_n\}$ is a Cauchy sequence in C. Since C is a closed in the Banach space E, so that there exists a point p in C such that $\lim_{n\to\infty} x_n = p$. Now $\lim_{n\to\infty} d(x_n, F(T)) = 0$ gives that $d(p, F(T)) = 0$. Since F is closed, $p \in F(T)$. This completes the proof. □

A mapping $T : C \to C$, where C is a subset of a normed space E, is said to satisfy Condition (A) [55] if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$, for all $r \in (0, 1)$, such that $\|x - Tx\| \geq f(d(x, F(T))$, for all $x \in C$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.

It is to be noted that Condition (A) is weaker than compactness of the domain C.

Applying Theorem 3.2.7, we obtain a strong convergence of the process (1.0.12) under Condition (A) as follows:

Theorem 3.2.8 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (1.0.12) and $F(T) \neq \emptyset$. Let T satisfy Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We proved in Lemma 3.2.5 that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.2.14}$$

From Condition (A) and (3.2.14), we get

$$\lim_{n\to\infty} f(d(x_n, F(T))) \le \lim_{n\to\infty} \|x_n - Tx_n\| = 0,$$

i.e., $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0,\infty) \to [0,\infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0,\infty)$, we have

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Therefore, by Theorem 3.2.7, the sequence $\{x_n\}$ converges strongly to a point of $F(T)$. The proof is completed. □

3.2.3 Applications

This section is devoted to some applications. Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and $T : C \to H$ a nonlinear operator. T is said to be:

(i) monotone if $\langle Tx - Ty, x - y\rangle \ge 0$ for all $x, y \in C$,

(ii) λ-strongly monotone if there exists a constant $\lambda > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge \lambda\|x - y\|^2$$

for all $x, y \in C$,

(iii) v-inverse strongly monotone (v–ism) if there exists a constant $v > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ge v\|Tx - Ty\|^2$$

for all $x, y \in C$.

Construction of fixed points of nonexpansive operators is an important subject in the theory of nonexpansive operators and has applications in a number of applied areas such as image recovery and signal processing (see, [13, 42, 68]. For instance, split feasibility problem of C and T (denoted by $SFP(C,T)$) is to find a point

$$x \text{ in } C \text{ such that } Tx \in Q, \tag{3.2.15}$$

where C is a closed convex subset of a Hilbert space H_1, Q is a closed convex subset of another Hilbert space H_2 and $T : H_1 \to H_2$ is a bounded linear operator. The $SFP(C,T)$ is said to be consistent if (3.2.15) has a solution. It is easy to see that $SFP(C,T)$ is consistent if and only if the following fixed point problem has a solution:

$$\text{find } x \in C \text{ such that } x = P_C(I - \gamma T^*(I - P_Q)T)x, \tag{3.2.16}$$

where P_C and P_Q are the orthogonal projections onto C and Q, respectively; $\gamma > 0$, and T^* is the adjoint of T. Note that for sufficient small $\gamma > 0$, the operator $P_C(I - \gamma T^*(I - P_Q)T)$ in (3.2.16) is nonexpansive.

3.2.3.1 Application to constrained minimization problems

Let C be a closed convex subset of a Hilbert space H, P_C the metric projection of H onto C and $T : C \to H$ a v-ism where $v > 0$ is a constant. It is well known that $P_C(I - \mu T)$ is nonexpansive operator provided that $\mu \in (0, 2v)$.

The algorithms for signal and image processing are often iterative constrained optimization processes designed to minimize a convex differentiable function T over a closed convex set C in H. It is well known that every L–Lipschitzian operator is $2/L$-ism.

Therefore, we have the following result which generates the sequence of vectors in the constrained or feasible set C which converges weakly to the optimal solution which minimizes T.

Theorem 3.2.9 *Let C be a closed convex subset of a Hilbert space H and T a convex and differentiable function on an open set D containing the set C. Assume that $\triangledown T$ is an L–Lipschitz operator on D, $\mu \in (0, 2/L)$ and minimizers of T relative to the set C exist. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)P_C(I - \mu \triangledown T)x_n + \alpha_n P_C(I - \mu \triangledown T)y_n, \\ y_n = (1-\beta_n)P_C(I - \mu \triangledown T)x_n + \beta_n P_C(I - \mu \triangledown T)z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n P_C(I - \mu \triangledown T)x_n, \; n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$. Then $\{x_n\}$ converges weakly to a minimizer of T.

3.2.3.2 Application to image deblurring problems

Let us consider the linear system: find $x \in C$ such that

$$Ax = b,$$

where $A : H \to H$ is bounded linear operator and $b \in H$ is fixed. An algorithm in Theorem 3.2.9 can be applied directly to solve

$$\min_x \|b - Ax\|_2, \tag{3.2.17}$$

by setting

$$T = \frac{1}{2}\|b - Ax\|_2^2.$$

Theorem 3.2.10 *Let H be a real Hilbert space. Let $A : H \to H$ be a bounded linear operator and $b \in H$. Let $\{x_n\}$ be a sequence generated by $C_1 = H$, $x_0 \in H$ and*

$$\begin{cases} x_{n+1} = (1-\alpha_n)\left(x_n - \mu A^T(Ax_n - b)\right) + \alpha_n \left(y_n - \mu A^T(Ay_n - b)\right), \\ y_n = (1-\beta_n)\left(x_n - \mu A^T(Ax_n - b)\right) + \beta_n \left(z_n - \mu A^T(Az_n - b)\right), \\ z_n = (1-\gamma_n)x_n + \gamma_n \left(x_n - \mu A^T(Ax_n - b)\right), \ n \in \mathbb{N}, \end{cases}$$

where $\mu \subset \left(0, \frac{2}{||A||_2^2}\right)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0,1)$. Then $\{x_n\}$ converges weakly to its solution.

The algorithm in Theorem 3.2.10 (implemented algorithm) can be used in solving image restoration problem

$$b = Ax + \upsilon.$$

Here b is the observed blurred and noisy image (degraded image), υ is an unknown gaussian noise, and A is a blurring matrix. The blurring matrix A is often ill–conditioned. The aim is to compute an approximation of the original image x. In the most case, the blur generally has much more significant effect than the noise, and thus, the emphasis is on removing the blur. Therefore, the original image x can be approximated by solving an equation (3.2.17). We call this kind of problem solving as image deblurring problem. An implemented algorithm is proposed in solving the image deblurring problem. Two kinds of image deblurring consists of Gaussian blur and motion blur are used to test the implemented algorithm.

The original grey and its degraded images on Figure 3.2.3 from a Gaussian blur of size 9×9, $\sigma = 4$ and the motion blur with len $= 21$, $\theta = 11$ respectively are used to test an implemented algorithm.

Figure 3.2.3: Data used in numerical experiments. The true grey image of size 336×252 and two types of its degraded image.

The parameters $\alpha_n, \beta_n, \gamma_n$ and μ on an implemented algorithm in solving the image deblurring problem is set as

$$\alpha_n = \frac{n}{n+1}, \beta_n = \frac{n}{\sqrt{4n^3+2}}, \gamma_n = \frac{n+1}{5n+3}, \mu = 1/\left(||A||_1||A||_\infty\right).$$

These parameters is called as the default choice of set parameter. The quality improvements of the reconstructed grey images sized 336×252 being used implemented algorithm are illustrated in Figure 3.2.4 and Figure 3.2.5.

Figure 3.2.4: The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the Gaussian blurred image on Figure 3.2.3.

Figure 3.2.5: The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the motion blurred image on Figure 3.2.3.

Next, we are also apply our algorithm in solving the color deblurring problems. The following RGB images illustrate example of blurring adjustment. The three independent deblurring problem consists of red green and blue deblurring channel which are solved with the default parameter.

Figure 3.2.6: Data used in numerical experiments. The true RGB image of size $336 \times 252 \times 3$, Degraded image with Gaussian blur of size 9×9 and $\sigma = 4$ and the motion blur image with len $= 21$ and $\theta = 11$.

The quality improvements of the reconstructed RGB images sized $336 \times 252 \times 3$ being used the implemented algorithm are also illustrated on Figure 3.2.7 and Figure 3.2.8.

Figure 3.2.7: The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the Gaussian blurred image on Figure 3.2.6.

Figure 3.2.8: The reconstructed images being 50th, 1000th, 10000th and 20000th used iterations of the Motion blurred image on Figure 3.2.6.

It can be seen that the restored images on Figures 3.2.4, 3.2.5, 3.2.7, 3.2.8 are clearly evident and we also essentially have the original image when the number of iteration is sufficient.

3.2.3.3 Application to split feasibility problems

Recall that a mapping T in a Hilbert space H is said to be averaged if T can be written as $(1-\alpha)I+\alpha S$, where $\alpha \in (0,1)$ and S is a nonexpansive map on H. Set

$$q(x) := \frac{1}{2}\|(T-P_QT)x\|, x \in C.$$

Consider the minimization problem

$$\text{find} \min_{x\in C} q(x).$$

By [5], the gradient of q is $\bigtriangledown q = T^*(I - P_Q)T$, where T^* is the adjoint of T. Since $I - P_Q$ is nonexpansive, it follows that $\bigtriangledown q$ is L-Lipschitzian with $L = \|T\|^2$. Therefore, $\bigtriangledown q$ is $1/L$ism and for any $0 < \mu < 2/L$, $I - \mu \bigtriangledown q$ is averaged. Therefore, the composition $P_C(I - \mu \bigtriangledown q)$ is also averaged. Set $T := P_C(I - \mu \bigtriangledown q)$. Note that the solution set of $SFP(C,T)$ is $F(T)$.

We now present an iterative process that can be used to find solutions of $SFP(C,T)$.

Theorem 3.2.11 *Assume that $SFP(C,T)$ is consistent. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0,1)$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)P_C(I - \mu \bigtriangledown q)x_n + \alpha_n P_C(I - \mu \bigtriangledown q)y_n, \\ y_n = (1-\beta_n)P_C(I - \mu \bigtriangledown q)x_n + \beta_n P_C(I - \mu \bigtriangledown q)z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n P_C(I - \mu \bigtriangledown q)x_n,\ n \in \mathbb{N}, \end{cases}$$

where $0 < \mu < 2/\|T\|^2$. Then $\{x_n\}$ converges weakly to a solution of $SFP(C,T)$.

Proof. Since $T := P_C(I - \mu \bigtriangledown q)$ is nonexpansive, the result follows from Theorem 3.2.6. □

3.3 New iterative method for nonlinear operators as concerns convex programming, image deblurring, and signal recovering problems

3.3.1 Main results

Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space $\mathcal{E}$ with $\mathcal{Q}_{\mathcal{K}}$ as a sunny nonexpansive retraction. We will represent the set of common fixed points of $\mathcal{S}$ and $\mathcal{T}$ by Ψ, that is, $\Psi := Fix(\mathcal{S}) \cap Fix(\mathcal{T})$. The following lemma is needed.

Lemma 3.3.1 *Let $\mathcal{K}$ be a nonempty closed convex subset of a Banach space $\mathcal{E}$ with $\mathcal{Q}_{\mathcal{K}}$ as the sunny nonexpansive retraction. Let $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{E}$ be quasi-nonexpansive mappings with $\Psi \neq \emptyset$ and let $\{\eta_n\}, \{\vartheta_n\}, \{\zeta_n\}$ be sequences in $(0,1)$ for all $n \in \mathbb{N}$. Define the sequence $\{u_n\}$ using Algorithm 1. Then, for each $\bar{u} \in \Psi$, $\lim_{n\to\infty} \|u_n - \bar{u}\|$ exists and*

$$\|w_n - \bar{u}\| \leq \|u_n - \bar{u}\|, \text{ and } \|z_n - \bar{u}\| \leq \|u_n - \bar{u}\|, \ \forall n \in \mathbb{N}. \tag{3.3.1}$$

Algorithm 1

---

initialization: $\eta_n, \vartheta_n, \zeta_n \in (0,1), u_1 \in \mathcal{K}$ and $n = 1$.
while stopping criterion not met **do**

$$\begin{aligned} w_n &= \mathcal{Q}_{\mathcal{K}}\left((1-\zeta_n)u_n + \zeta_n \mathcal{S} u_n\right), \\ z_n &= \mathcal{Q}_{\mathcal{K}}\left((1-\vartheta_n)\mathcal{T} u_n + \vartheta_n \mathcal{S} w_n\right), \\ u_{n+1} &= \mathcal{Q}_{\mathcal{K}}\left((1-\eta_n)\mathcal{S} z_n + \eta_n \mathcal{T} w_n\right). \end{aligned}$$

end

Proof. Let $\bar{u} \in \Psi$. Then, for each $n \geq 1$, we have

$$\begin{aligned} \|w_n - \bar{u}\| &= \|\mathcal{Q}_{\mathcal{K}}((1-\zeta_n)u_n + \zeta_n \mathcal{S} u_n) - \bar{u}\| \\ &\leq \|(1-\zeta_n)(u_n - \bar{u}) + \zeta_n(\mathcal{S} u_n - \bar{u})\| \end{aligned}$$

$$\leq (1-\zeta_n)\|u_n - \bar{u}\| + \zeta_n\|\mathcal{S}u_n - \bar{u}\|$$
$$\leq (1-\zeta_n)\|u_n - \bar{u}\| + \zeta_n\|u_n - \bar{u}\|$$
$$= \|u_n - \bar{u}\|, \tag{3.3.2}$$

$$\begin{aligned}\|z_n - \bar{u}\| &= \|\mathcal{Q}_{\mathcal{K}}((1-\vartheta_n)\mathcal{T}u_n + \vartheta_n\mathcal{S}w_n) - \bar{u}\| \\ &\leq \|(1-\vartheta_n)(\mathcal{T}u_n - \bar{u}) + \vartheta_n(\mathcal{S}w_n - \bar{u})\| \\ &\leq (1-\vartheta_n)\|\mathcal{T}u_n - \bar{u}\| + \vartheta_n\|\mathcal{S}w_n - \bar{u}\| \\ &\leq (1-\vartheta_n)\|u_n - \bar{u}\| + \vartheta_n\|w_n - \bar{u}\| \\ &\leq (1-\vartheta_n)\|u_n - \bar{u}\| + \vartheta_n\|u_n - \bar{u}\| \\ &= \|u_n - \bar{u}\| \end{aligned} \tag{3.3.3}$$

and

$$\begin{aligned}\|u_{n+1} - \bar{u}\| &= \|\mathcal{Q}_{\mathcal{K}}((1-\eta_n)\mathcal{S}z_n + \eta_n\mathcal{T}w_n) - \bar{u}\| \\ &\leq \|(1-\eta_n)(\mathcal{S}z_n - \bar{u}) + \eta_n(\mathcal{T}w_n - \bar{u})\| \\ &\leq (1-\eta_n)\|\mathcal{S}z_n - \bar{u}\| + \eta_n\|\mathcal{T}w_n - \bar{u}\| \\ &\leq (1-\eta_n)\|z_n - \bar{u}\| + \eta_n\|w_n - \bar{u}\| \\ &\leq (1-\eta_n)\|u_n - \bar{u}\| + \eta_n\|u_n - \bar{u}\| \\ &= \|u_n - \bar{u}\|. \end{aligned} \tag{3.3.4}$$

Therefore,

$$\|u_{n+1} - \bar{u}\| \leq \|u_n - \bar{u}\| \leq ... \leq \|u_1 - \bar{u}\|, \ \forall n \in \mathbb{N}. \tag{3.3.5}$$

Since $\{\|u_n - \bar{u}\|\}$ is monotonically decreasing, we have that the sequence $\{\|u_n - \bar{u}\|\}$ is convergent. □

Theorem 3.3.2 *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{E}$ with $\mathcal{Q}_{\mathcal{K}}$ as the sunny nonexpansive retraction. Let $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{E}$ be quasi-nonexpansive mappings with $\Psi \neq \emptyset$. Let $\{\eta_n\}$,$\{\vartheta_n\}$, $\{\zeta_n\}$ be sequences of real numbers such that $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. From an arbitrary $u_1 \in \mathcal{K}$, $\mathcal{P}_\Psi(u_1) = u_*$, define the sequence $\{u_n\}$ by Algorithm 1. Then, we have the following:*

(i) *$\{u_n\}$ is in a closed convex bounded set $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$, where λ is a constant in $(0, \infty)$ such that $\|u_1 - u_*\| \leq \lambda$.*

(ii) *If $\mathcal{S}$ is uniformly continuous, then $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$ and $\lim_{n\to\infty} \|u_n - \mathcal{T}u_n\| = 0$.*

(iii) *If $\mathcal{E}$ fulfills the Opial's condition and $\mathcal{I} - \mathcal{S}$ and $\mathcal{I} - \mathcal{T}$ are demiclosed at 0, then $\{u_n\}$ converges weakly to an element of $\Psi \cap \mathcal{B}_\lambda[u_*]$.*

Proof. (i) Since $u_* \in \Psi$, from (3.3.5), we obtain

$$\|u_{n+1} - u_*\| \leq \|u_n - u_*\| \leq \ldots \leq \|u_1 - u_*\| \leq \lambda, \ \forall n \in \mathbb{N}. \tag{3.3.6}$$

Therefore, $\{u_n\}$ is in the closed convex bounded set $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$.

(ii) Suppose that $\mathcal{S}$ is uniformly continuous. Using Lemma 3.3.1, we get that $\{u_n\}$,$\{z_n\}$ and $\{w_n\}$ are in $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$, and hence, from (3.3.1), we obtain $\|\mathcal{T}u_n - u_*\| \leq \lambda$, $\|\mathcal{T}w_n - u_*\| \leq \lambda$, $\|\mathcal{S}u_n - u_*\| \leq \lambda$, $\|\mathcal{S}w_n - u_*\| \leq \lambda$ and $\|\mathcal{S}z_n - u_*\| \leq \lambda$, $\forall n \in \mathbb{N}$. Using Lemma 2.4.15 for $p = 2$ and $R = \lambda$, from the relations in Algorithm 1, we obtain

$$\begin{aligned}
\|w_n - u_*\|^2 &\leq (1 - \zeta_n)\|u_n - u_*\|^2 + \zeta_n\|\mathcal{S}u_n - u_*\|^2 \\
&\quad - \zeta_n(1 - \zeta_n)\varphi(\|u_n - \mathcal{S}u_n\|) \\
&\leq (1 - \zeta_n)\|u_n - u_*\|^2 + \zeta_n\|u_n - u_*\|^2 \\
&\quad - \zeta_n(1 - \zeta_n)\varphi(\|u_n - \mathcal{S}u_n\|) \\
&= \|u_n - u_*\|^2 - \zeta_n(1 - \zeta_n)\varphi(\|u_n - \mathcal{S}u_n\|)
\end{aligned} \tag{3.3.7}$$

and

$$\begin{aligned}\|z_n - u_*\|^2 &\leq (1-\vartheta_n)\|u_n - u_*\|^2 + \vartheta_n\|w_n - u_*\|^2 \\ &\quad - \vartheta_n(1-\vartheta_n)\varphi(\|\mathcal{T}u_n - \mathcal{S}w_n\|) \\ &\leq (1-\vartheta_n)\|u_n - u_*\|^2 + \vartheta_n\|u_n - u_*\|^2 \\ &\quad - \vartheta_n(1-\vartheta_n)\varphi(\|\mathcal{T}u_n - \mathcal{S}w_n\|) \\ &= \|u_n - u_*\|^2 - \vartheta_n(1-\vartheta_n)\varphi(\|\mathcal{T}u_n - \mathcal{S}w_n\|). \qquad (3.3.8)\end{aligned}$$

Using (3.3.7), (3.3.8) and Lemma 2.4.15, we have

$$\begin{aligned}\|u_{n+1} - u_*\|^2 &\leq \|(1-\eta_n)(\mathcal{S}z_n - u_*) + \eta_n(\mathcal{T}w_n - u_*)\|^2 \\ &\leq (1-\eta_n)\|\mathcal{S}z_n - u_*\|^2 + \eta_n\|\mathcal{T}w_n - u_*\|^2 \\ &\quad - \eta_n(1-\eta_n)\varphi(\|\mathcal{S}z_n - \mathcal{T}w_n\|) \\ &\leq (1-\eta_n)\|z_n - u_*\|^2 + \eta_n\|w_n - u_*\|^2 \\ &\quad - \eta_n(1-\eta_n)\varphi(\|\mathcal{S}z_n - \mathcal{T}w_n\|) \\ &\leq (1-\eta_n)(\|u_n - u_*\|^2 - \vartheta_n(1-\vartheta_n)\varphi(\|\mathcal{T}u_n - \mathcal{S}w_n\|)) \\ &\quad + \eta_n(\|u_n - u_*\|^2 - \zeta_n(1-\zeta_n)\varphi(\|u_n - \mathcal{S}u_n\|)) \\ &\quad - \eta_n(1-\eta_n)\varphi(\|\mathcal{S}z_n - \mathcal{T}w_n\|) \\ &= \|u_n - u_*\|^2 - (1-\eta_n)\vartheta_n(1-\vartheta_n)\varphi(\|\mathcal{T}u_n - \mathcal{S}w_n\|) \\ &\quad - \eta_n\zeta_n(1-\zeta_n)\varphi(\|u_n - \mathcal{S}u_n\|) \\ &\quad - \eta_n(1-\eta_n)\varphi(\|\mathcal{S}z_n - \mathcal{T}w_n\|). \qquad (3.3.9)\end{aligned}$$

From (3.3.9), we have the following important two inequalities.

$$(1-\eta_n)\vartheta_n(1-\vartheta_n)\varphi(\|\mathcal{T}u_n - \mathcal{S}w_n\|) \leq \|u_n - u_*\|^2 - \|u_{n+1} - u_*\|^2 \qquad (3.3.10)$$

and

$$\eta_n\zeta_n(1-\zeta_n)\varphi(\|u_n - \mathcal{S}u_n\|) \leq \|u_n - u_*\|^2 - \|u_{n+1} - u_*\|^2. \qquad (3.3.11)$$

Note that $(1-\hat{c}_1)c_2(1-\hat{c}_2) \leq (1-\eta_n)\vartheta_n(1-\vartheta_n)$ and $c_1c_3(1-\hat{c}_3) \leq \eta_n\zeta_n(1-\zeta_n)$.

Using (3.3.10) and (3.3.11), we obtain

$$(1-\hat{c}_1)c_2(1-\hat{c}_2)\sum_{i=1}^{n}\varphi(\|\mathcal{T}u_i - \mathcal{S}w_i\|) \leq \|u_1 - u_*\|^2 - \|u_{n+1} - u_*\|^2, \ \forall n \in \mathbb{N} \tag{3.3.12}$$

and

$$c_1c_3(1-\hat{c}_3)\sum_{i=1}^{n}\varphi(\|u_i - \mathcal{S}u_i\|) \leq \|u_1 - u_*\|^2 - \|u_{n+1} - u_*\|^2, \ \forall n \in \mathbb{N}. \tag{3.3.13}$$

It follows from (3.3.12) and (3.3.13), we obtain

$$\sum_{n=1}^{\infty}\varphi(\|\mathcal{T}u_n - \mathcal{S}w_n\|) < \infty \tag{3.3.14}$$

and

$$\sum_{n=1}^{\infty}\varphi(\|u_n - \mathcal{S}u_n\|) < \infty. \tag{3.3.15}$$

Using (3.3.14) and (3.3.15), we have

$$\lim_{n\to\infty}\|\mathcal{T}u_n - \mathcal{S}w_n\| = 0 \tag{3.3.16}$$

and

$$\lim_{n\to\infty}\|u_n - \mathcal{S}u_n\| = 0. \tag{3.3.17}$$

In addition, using (3.3.17), we have

$$\begin{aligned}\|w_n - u_n\| &= \|\mathcal{Q}_{\mathcal{K}}((1-\zeta_n)u_n + \zeta_n\mathcal{S}u_n) - \mathcal{Q}_{\mathcal{K}}(u_n)\|,\\ &\leq \|\mathcal{S}u_n - u_n\|\\ &\to 0 \ (as \ \ n\to\infty).\end{aligned} \tag{3.3.18}$$

Since $\mathcal{S}$ is uniformly continuous, it follows from Lemma 2.4.16 that

$$\lim_{n\to\infty} \|w_n - \mathcal{S}w_n\| = 0. \tag{3.3.19}$$

Thus from (3.3.16), (3.3.17), (3.3.18) and (3.3.19), we have

$$\begin{aligned}
\|u_n - \mathcal{T}u_n\| &\leq \|u_n - \mathcal{S}u_n\| + \|\mathcal{S}u_n - \mathcal{T}u_n\| \\
&\leq \|u_n - \mathcal{S}u_n\| + \|\mathcal{S}u_n - \mathcal{S}w_n\| + \|\mathcal{S}w_n - \mathcal{T}u_n\| \\
&\leq \|u_n - \mathcal{S}u_n\| + \|\mathcal{S}u_n - u_n\| + \|u_n - \mathcal{S}w_n\| \\
&\quad + \|\mathcal{S}w_n - \mathcal{T}u_n\| \\
&\leq \|u_n - \mathcal{S}u_n\| + \|\mathcal{S}u_n - u_n\| + \|u_n - w_n\| \\
&\quad + \|w_n - \mathcal{S}w_n\| + \|\mathcal{S}w_n - \mathcal{T}u_n\| \\
&\quad \to 0 \ (as \ \ n \to \infty).
\end{aligned}$$

(iii) By assumption, $\mathcal{E}$ satisfies the Opial's condition. Let $w^* \in \Psi$ such that $w^* \in \mathcal{B}_\lambda[u_*] \cap \mathcal{K}$. From Lemma 3.3.1, we have $\lim_{n\to\infty} \|u_n - w^*\|$ exists. Suppose there are two subsequences $\{u_{n_q}\}$ and $\{u_{m_l}\}$ which converge to two distinct points u^* and v^* in $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$, respectively. Then, since both $\mathcal{I} - \mathcal{S}$ and $\mathcal{I} - \mathcal{T}$ have the demiclosed property at 0, we have $\mathcal{S}u^* = \mathcal{T}u^* = u^*$ and $\mathcal{S}v^* = \mathcal{T}v^* = v^*$. Moreover, using the Opial's condition, we have

$$\lim_{n\to\infty} \|u_n - u^*\| = \lim_{q\to\infty} \|u_{n_q} - u^*\| < \lim_{l\to\infty} \|u_{m_l} - v^*\| = \lim_{n\to\infty} \|u_n - v^*\|.$$

Similarly, we obtain

$$\lim_{n\to\infty} \|u_n - v^*\| < \lim_{n\to\infty} \|u_n - u^*\|$$

which is a contradiction. Therefore, $u^* = v^*$. Hence, the sequence $\{u_n\}$ converges weakly to an element of $\Psi \cap \mathcal{B}_\lambda[u_*] \cap \mathcal{K}$. $\square$

Since every nonexpansive mapping is uniformly continuous. By using the same ideas and techniques as in Theorem 3.3.2 and using Lemma 2.4.17, we can state the following result without proofs.

Theorem 3.3.3 *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{E}$ with $\mathcal{Q}_{\mathcal{K}}$ as the sunny nonexpansive retraction. Let $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{E}$ be nonexpansive mappings with $\psi \neq \emptyset$. Let $\{\eta_n\}$,$\{\vartheta_n\}$, $\{\zeta_n\}$ be sequences of real numbers such that $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. From an arbitrary $u_1 \in \mathcal{K}$, $\mathcal{P}_{\Psi}(u_1) = u_*$, define the sequence $\{u_n\}$ by Algorithm 1. Then, we have the following:*

(i) $\{u_n\}$ *is in a closed convex bounded set $\mathcal{B}_{\lambda}[u_*] \cap \mathcal{K}$ where λ is a constant in $(0, \infty)$ such that $\|u_1 - u_*\| \leq \lambda$.*

(ii) $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$ *and* $\lim_{n\to\infty} \|u_n - \mathcal{T}u_n\| = 0$.

(iii) If $\mathcal{E}$ fulfills the Opial's condition, then $\{u_n\}$ converges weakly to an element of $\Psi \cap \mathcal{B}_{\lambda}[u_*]$.

If $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive self mappings on a nonempty closed convex subset $\mathcal{K}$ of a real Hilbert space $\mathcal{H}$. As a result of Theorem 3.3.2, we can get the following result.

Corollary 3.3.4 *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, let $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ be nonexpansive mappings with $\Psi \neq \emptyset$ and let $\{\eta_n\}$,$\{\vartheta_n\}$ and $\{\zeta_n\}$ be sequences of real numbers, for which $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. From an arbitrary $u_1 \in \mathcal{K}$, define the sequence $\{u_n\}$ by :*

$$\begin{cases} w_n = (1 - \zeta_n)u_n + \zeta_n \mathcal{S}u_n \\ z_n = (1 - \vartheta_n)\mathcal{T}u_n + \vartheta_n \mathcal{S}w_n \\ u_{n+1} = (1 - \eta_n)\mathcal{S}z_n + \eta_n \mathcal{T}w_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{3.3.20}$$

Then, $\{u_n\}$ converges weakly to an element of Ψ.

3.3.2 Applications

In the following part, we study common zeros of accretive operators, convexly constrained least square problems, and convex minimization problems using the methods outlined above.

3.3.2.1 Application to common zeros of accretive operators

Setting $\mathcal{S} = J_\mu^{\mathcal{A}}$ and $\mathcal{T} = J_\mu^{\mathcal{B}}$, using (3.3.20), we derive its convergence analysis for finding solutions (1.0.13).

Theorem 3.3.5 *Let $\mathcal{K}$ be a nonempty closed convex subset of a real uniformly convex Banach space $\mathcal{E}$ satisfying the opial's condition. Let $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{K} \to 2^{\mathcal{E}}$, $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subseteq \mathcal{K} \to 2^{\mathcal{E}}$ be accretive operators such that $\overline{\mathcal{D}(\mathcal{A})} \subseteq \mathcal{K} \subseteq \cap_{\mu>0}\mathcal{R}(\mathcal{I}+\mu\mathcal{A})$, $\overline{\mathcal{D}(\mathcal{B})} \subseteq \mathcal{K} \subseteq \cap_{\mu>0}\mathcal{R}(\mathcal{I}+\mu\mathcal{B})$ and $\mathcal{A}^{-1}(0) \cap \mathcal{B}^{-1}(0) \neq \emptyset$. Let $\{\eta_n\}$, $\{\vartheta_n\}$, $\{\zeta_n\}$ be sequences of real numbers such that $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. Let $\mu > 0$, $u_1 \in \mathcal{K}$ and $\mathcal{P}_{\mathcal{A}^{-1}(0)\cap\mathcal{B}^{-1}(0)}(u_1) = u_*$. Let $\{u_n\}$ be defined by*

$$\begin{cases} w_n = (1-\zeta_n)u_n + \zeta_n J_\mu^{\mathcal{A}} u_n, \\ z_n = (1-\vartheta_n) J_\mu^{\mathcal{B}} u_n + \vartheta_n J_\mu^{\mathcal{A}} w_n, \\ u_{n+1} = (1-\eta_n) J_\mu^{\mathcal{A}} z_n + \eta_n J_\mu^{\mathcal{B}} w_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{3.3.21}$$

Then, we have the following:

(i) $\{u_n\}$ is in a closed convex bounded set $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$, where λ is a constant in $(0, \infty)$ such that $\|u_1 - u_*\| \leq \lambda$.

(ii) $\lim_{n\to\infty} \|u_n - J_\mu^{\mathcal{A}} u_n\| = 0$ and $\lim_{n\to\infty} \|u_n - J_\mu^{\mathcal{B}} u_n\| = 0$.

(iii) $\{u_n\}$ converges weakly to an element of $\mathcal{A}^{-1}(0) \cap \mathcal{B}^{-1}(0) \cap \mathcal{B}_\lambda[u_*]$.

Proof. By assumption $\overline{\mathcal{D}(\mathcal{A})} \subseteq \mathcal{K} \subseteq \cap_{\mu>0}\mathcal{R}(\mathcal{I} + \mu\mathcal{A})$, we known that $J_\mu^{\mathcal{A}}, J_\mu^{\mathcal{B}}$:

$\mathcal{K} \to \mathcal{K}$ be nonexpansive. Note that $\mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B}) \subseteq \mathcal{K}$ and hence

$$\begin{aligned} u_* \in \mathcal{A}^{-1}(0) \cap \mathcal{B}^{-1}(0) &\Rightarrow u_* \in \mathcal{D}(\mathcal{A}) \cap \mathcal{D}(\mathcal{B}) \; with \; 0 \in \mathcal{A}u_* \; and \; 0 \in \mathcal{B}u_* \\ &\Rightarrow u_* \in \mathcal{K} \; with \; J_\mu^{\mathcal{A}} u_* = u_* \; and \; J_\mu^{\mathcal{B}} u_* = u_* \\ &\Rightarrow u_* \in Fix(J_\mu^{\mathcal{A}}, J_\mu^{\mathcal{B}}) \cap \mathcal{K}. \end{aligned}$$

Setting $\mathcal{S} = J_\mu^{\mathcal{A}}$ and $\mathcal{T} = J_\mu^{\mathcal{B}}$. Hence, Theorem 3.3.5 is the same way as Theorem 3.3.3. □

3.3.2.2 Application to the convexly constrained least square problems

Let $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$, and $y, z \in \mathcal{H}$. Define $\varphi, \psi : \mathcal{H} \to \mathbb{R}$ by

$$\varphi(u) = \frac{1}{2}\|\mathcal{A}u - y\|^2 \text{ and } \psi(u) = \frac{1}{2}\|\mathcal{B}u - z\|^2$$

for all $u \in \mathcal{H}$, where $\mathcal{H}$ is a real Hilbert space. Let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{H}$. The objective is to find $b \in \mathcal{K}$ such that

$$b \in \underset{u \in \mathcal{K}}{argmin}\, \varphi(u) \cap \underset{u \in \mathcal{K}}{argmin}\, \psi(u), \tag{3.3.22}$$

where $\underset{u \in \mathcal{K}}{argmin}\, \varphi(u) := \{u_* \in \mathcal{K} : \varphi(u_*) = \underset{u \in \mathcal{K}}{inf}\, \varphi(u)\}$.

Theorem 3.3.3 is applied to find common solutions to two convexly constrained least square problems.

Proposition 3.3.6 *[1]. Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ with the adjoint $\mathcal{A}^*$ and $y \in \mathcal{H}$. Let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $b \in \mathcal{H}$ and $\delta \in (0, \infty)$. Then, the following statements are equivalent:*

(i) *b solves the following problem:*

$$\min_{u\in\mathcal{K}} \frac{1}{2}\|\mathcal{A}u - y\|^2.$$

(ii) $b = \mathcal{P}_{\mathcal{K}}(b - \delta\mathcal{A}^*(\mathcal{A}b - y))$.

(iii) $\langle \mathcal{A}v - \mathcal{A}b, y - \mathcal{A}b\rangle \leq 0$, *for all* $v \in \mathcal{K}$.

Theorem 3.3.7 *Let* $\mathcal{K}$ *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$, $y, z \in \mathcal{H}$ *and* $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$ *such that the solution set of the problem in (3.3.22) is nonempty. Let* $\{\eta_n\}$, $\{\vartheta_n\}$ *and* $\{\zeta_n\}$ *be sequences of real numbers such that* $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ *for all* $n \in \mathbb{N}$. *Let* $u_1 \in \mathcal{H}$, $\mathcal{P}_{argmin_{u\in\mathcal{K}}\varphi(u) \cap argmin_{u\in\mathcal{K}}\psi(u)}(u_1) = u_*$ *and* $\delta \in (0, 2min\{\frac{1}{\|\mathcal{A}\|^2}, \frac{1}{\|\mathcal{B}\|^2}\})$. *From an arbitrary* $u_1 \in \mathcal{K}$, *define the sequence* $\{u_n\}$ *by*

$$\begin{cases} w_n = (1-\zeta_n)u_n + \zeta_n \mathcal{S}u_n, \\ z_n = (1-\vartheta_n)\mathcal{T}u_n + \vartheta_n\mathcal{S}w_n, \\ u_{n+1} = (1-\eta_n)\mathcal{S}z_n + \eta_n\mathcal{T}w_n, \forall n \in \mathbb{N}, \end{cases} \tag{3.3.23}$$

where $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ *defined by* $\mathcal{S}u = \mathcal{P}_{\mathcal{K}}(u - \delta\mathcal{A}^*(\mathcal{A}u - y))$ *and* $\mathcal{T}u = \mathcal{P}_{\mathcal{K}}(u - \delta\mathcal{B}^*(\mathcal{B}u - z))$ *for all* $u \in \mathcal{K}$. *Then, we have the following:*

(i) $\{u_n\}$ *is in the closed ball* $\mathcal{B}_\lambda[u_*]$, *where* λ *is a constant in* $(0, \infty)$ *such that* $\|u_1 - u_*\| \leq \lambda$.

(ii) $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$ *and* $\lim_{n\to\infty} \|u_n - \mathcal{T}u_n\| = 0$.

(iii) $\{u_n\}$ *converges weakly to an element of*

$$argmin_{u\in\mathcal{K}}\, \varphi(u) \cap argmin_{u\in\mathcal{K}}\, \psi(u) \cap d\, \mathcal{B}_\lambda[u_*].$$

Proof. Note that: $\nabla\varphi(u) = \mathcal{A}^*(\mathcal{A}u - y)$, for all $u \in \mathcal{H}$; we obtain that $\|\nabla\varphi(u) - \nabla\varphi(v)\| = \|\mathcal{A}^*(\mathcal{A}u - y) - \mathcal{A}^*(\mathcal{A}v - y)\| \leq \|\mathcal{A}\|^2\|u - v\|$, for all $u, v \in \mathcal{H}$, Thus, $\nabla\varphi(u)$ is $\frac{1}{\|\mathcal{A}\|^2}$-ism and hence $(\mathcal{I} - \delta\nabla\varphi)$ is nonexpansive from $\mathcal{K}$ into $\mathcal{H}$ for $\sigma \in (0, \frac{2}{\|\mathcal{A}\|^2})$. Therefore, $\mathcal{S} = \mathcal{P}_{\mathcal{K}}(\mathcal{I} - \sigma\nabla\varphi)$ and $\mathcal{T} = \mathcal{P}_{\mathcal{K}}(\mathcal{I} - \tau\nabla\varphi))$ are nonexpansive mappings from $\mathcal{K}$ into itself for $\sigma \in (0, \frac{2}{\|\mathcal{A}\|^2})$ and $\tau \in (0, \frac{2}{\|\mathcal{B}\|^2})$, respectively. Thus, Theorem 3.3.7 follows from Theorem 3.3.3. □

3.3.3 Application to the convex minimization problems

Let $\mathcal{H}$ be a Hilbert space and let $g_1, g_2 : \mathcal{H} \to (-\infty, \infty]$ be proper convex and lower–semicontinuous functions. We focus on the problem of finding $x \in \mathcal{H}$ in such a form that

$$x \in \partial g_1^{-1}(0) \cap g_2^{-1}(0). \tag{3.3.24}$$

Note that $J_\mu^{\partial g_1} = prox_{\mu g_1}$. We provide an application to common solutions for convex programming problems in a Hilbert space.

Theorem 3.3.8 *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Let $g_1, g_2 \in \Gamma_0\mathcal{H}$ such that the solution set of the problem in (3.3.24) is nonempty. Let $\{\eta_n\}$, $\{\vartheta_n\}$ and $\{\zeta_n\}$ be sequences of real numbers such that $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. Let $\mu > 0$, $u_1 \in \mathcal{H}$ and $\mathcal{P}_{\partial g_1^{-1}(0) \cap g_2^{-1}(0)}(u_1) = u_*$. From an arbitrary $u_1 \in \mathcal{K}$, define the sequence $\{u_n\}$ by*

$$\begin{cases} w_n = (1 - \zeta_n)u_n + \zeta_n prox_{\mu g_1}(u_n), \\ z_n = (1 - \vartheta_n)prox_{\mu g_2}(u_n) + \vartheta_n prox_{\mu g_1}(w_n), \\ u_{n+1} = (1 - \eta_n)prox_{\mu g_1}(z_n) + \eta_n prox_{\mu g_2}(w_n), \ \forall n \in \mathbb{N}. \end{cases} \tag{3.3.25}$$

Then, we have the following:

(i) $\{u_n\}$ is in the closed ball $\mathcal{B}_\lambda[u_*]$ where λ is a constant in $(0, \infty)$ such that $\|u_1 - u_*\| \leq \lambda$.

(ii) $\lim_{n\to\infty} \|u_n - prox_{\mu g_1}(u_n)\| = 0$ and $\lim_{n\to\infty} \|u_n - prox_{\mu g_2}(u_n)\| = 0$.

(iii) $\{u_n\}$ converges weakly to an element of $\partial g_1^{-1}(0) \cap g_2^{-1}(0) \cap \mathcal{B}_\lambda[u_*]$.

Proof. Using Lemma 2.4.14, we have ∂g_1 is maximal monotone. We see that $\mathcal{R}(\mathcal{I} + \mu\partial f) = \mathcal{H}$, by the maximal monotonicity of ∂g_1. It follows that $J_\mu^{\partial g_1} = prox_{\mu g_1} : \mathcal{H} \to \mathcal{H}$ is nonexpansive. Similarly, $J_\mu^{\partial g_2} = prox_{\mu g_2} : \mathcal{H} \to \mathcal{H}$ is nonexpansive. Thus, Theorem 3.3.8 follows from Theorem 3.3.3. □

3.3.4 Numerical experiments in real world problems

In this section, some real world problems such as differential problems, image deblurring and signal recovering problems are used to illustrate the effective of our algorithm.

3.3.4.1 Differential problems

Let consider the following simple and well known one-dimensional heat equation with Dirichlet boundary conditions and initial data,

$$\begin{aligned} u_t = \beta u_{xx} + f(x,t), &\quad 0 < x < l, \quad t > 0. \\ u(x,0) = u_0(x), &\quad 0 < x < l, \\ u(0,t) = \gamma_1(t), \quad u(l,t) = \gamma_2(t), &\quad t > 0, \end{aligned} \tag{3.3.26}$$

where β is constant, $u(x,t)$ represents the temperature at point (x,t) and $f(x,t)$, $\gamma_1(t)$, $\gamma_2(t)$ are sufficiently smooth functions. Below, we use the notations u_i^n and $(u_{xx})_i^n$ to represent the numerical approximations of $u(x_i, t^n)$ and $u_{xx}(x_i, t^n)$ and $t^n = n\Delta t$ where Δt denotes the temporal mesh size. A set of schemes in solving problem (3.3.26) is based on the following well–known Crank–Nikolson type of scheme [70, 71],

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = \frac{\beta}{2}\left[(u_{xx})_i^{n+1} + (u_{xx})_i^n\right] + f_i^{n+1/2}, \quad i = 2, \ldots, N-1 \tag{3.3.27}$$

with initial data

$$u_i^0 = u^0(x_i), \quad i = 2, \ldots, N-1 \tag{3.3.28}$$

and Dirichlet boundary conditions

$$u_1^{n+1} = \gamma_1(t^{n+1}), \quad u_N^{n+1} = \gamma_2(t^{n+1}). \tag{3.3.29}$$

To approximate term of $(u_{xx})_i^k, k = n, n+1$, we use the standard centered discretization with space. The matrix form of second-order finite difference scheme (FDS) in solving heat problem (3.3.26) can be written as

$$A\mathbf{u}^{n+1} = \mathbf{G}^n, \tag{3.3.30}$$

where $\mathbf{G}^n = B\mathbf{u}^n + \mathbf{f}^{n+1/2}$,

$$A = \begin{bmatrix} 1+\eta & -\frac{\eta}{2} & & & \\ -\frac{\eta}{2} & 1+\eta & -\frac{\eta}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{\eta}{2} & 1+\eta & -\frac{\eta}{2} \\ & & & -\frac{\eta}{2} & 1+\eta \end{bmatrix}, \quad B = \begin{bmatrix} 1-\eta & \frac{\eta}{2} & & & \\ \frac{\eta}{2} & 1-\eta & \frac{\eta}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{\eta}{2} & 1-\eta & \frac{\eta}{2} \\ & & & \frac{\eta}{2} & 1-\eta \end{bmatrix},$$

$$\mathbf{u}^n = \begin{bmatrix} u_2^k \\ u_3^k \\ \vdots \\ u_{N-2}^k \\ u_{N-1}^k \end{bmatrix}, \quad \mathbf{f}^{n+1/2} = \begin{bmatrix} \frac{\eta}{2}\gamma_1^{n+1/2} + \Delta t f_2^{n+1/2} \\ \Delta t f_3^{n+1/2} \\ \vdots \\ \Delta t f_{N-2}^{n+1/2} \\ \frac{\eta}{2}\gamma_2^{n+1/2} + \Delta t f_{N-1}^{n+1/2} \end{bmatrix},$$

$\eta = \beta\Delta t/(\Delta x^2)$, $\gamma_i^{n+1/2} = \gamma_i(t^{n+1/2}), i = 1, 2$ and $f_i^{n+1/2} = f_i(t^{n+1/2})$, $i = 2, \ldots, N-1$.

From equation (3.3.30), matrix A is square and symmetric positive definite. Traditionally iterative methods have been presented in solving the solution of linear systems (3.3.30). The well-known weight Jacobi (WJ) and successive over relaxation (SOR) methods [23, 70] are chosen to exemplify here (see on Table 3.3.1).

Linear system	Iterative method	Specific name
$A\mathbf{u}^{n+1} = \mathbf{G}^n$	$D\mathbf{u}^{(n+1,s+1)} = (D - \omega A)\,\mathbf{u}^{(n+1,s)} + \omega\mathbf{G}^n$	WJ
	$(D - \omega L)\,\mathbf{u}^{(n+1,s+1)} = \Big((D - \omega L) - \omega A\Big)\mathbf{u}^{(n+1,s)} + \omega\mathbf{G}^n$	SOR

Table 3.3.1: The specific name of WJ and SOR in solving linear system (3.3.30).

And ω is weight parameter, D is the diagonal part of matrix A and L is the lower triangular part of matrix $D - A$, respectively. For stability of WJ and SOR method in solving linear system (3.3.30) generates from the discretization of the consideration problem (3.3.26), the step sizes of time play an important role of the stability needed. The discussion on the stability of WJ and SOR in solving linear system (3.3.30) can be found in [23, 70].

Let consider the linear system

$$A\mathbf{u} = \mathbf{G} \tag{3.3.31}$$

where $A : \mathbb{R}^l \to \mathbb{R}^l$ is linear and positive operator and $\mathbf{u}, \mathbf{G} \in \mathbb{R}^l$. To find the solution of linear system (3.3.31), we manipulate this linear system into the form of fixed point equation $\mathbf{u} = \mathcal{T}(\mathbf{u})$ (see on Table 3.3.2). For example, the well-known weight Jacobi (WJ), successive over relaxation (SOR) and Gauss-Seidel (GS, SOR with $\omega = 1$) methods [23, 70, 71] present the linear system (3.3.31) into the form of fixed point equation as $\mathcal{T}^{\mathrm{WJ}}(\mathbf{u}) = \mathbf{u}$, $\mathcal{T}^{\mathrm{SOR}}(\mathbf{u}) = \mathbf{u}$ and $\mathcal{T}^{\mathrm{GS}}(\mathbf{u}) = \mathbf{u}$ respectively.

Linear system	Fixed point mapping $\mathcal{T}(\mathbf{u})$
$A\mathbf{u} = \mathbf{G}$	$\mathcal{T}^{\mathrm{WJ}}(\mathbf{u}) = (I - \omega D^{-1}A)\,\mathbf{u} + \omega D^{-1}\mathbf{G}$
	$\mathcal{T}^{\mathrm{SOR}}(\mathbf{u}) = \left(I - \omega\,(D - \omega L)^{-1} A\right)\mathbf{u} + \omega\,(D - \omega L)^{-1}\,\mathbf{G}$

Table 3.3.2: The different way of rearranging linear systems (3.3.31) into the form $\mathbf{u} = T(\mathbf{u})$.

Let $\mathcal{T}\mathbf{u} = \mathcal{S}\mathbf{u} + \mathbf{c}$, where $\mathbf{u}, \mathbf{c} \in \mathcal{K}$ and $\mathcal{S}$ is a self mapping on a nonempty closed convex subset $\mathcal{K}$ of a uniformly convex Banach space $\mathcal{E}$ with $\| \mathcal{S} \| < 1$ then $\mathcal{T}$ is a nonexpansive mapping on $\mathcal{K}$. Indeed, for all $\mathbf{u}, \mathbf{v} \in \mathcal{K}$ and $n \geq 1$, we have

$$\| \mathcal{T}\mathbf{u} - \mathcal{T}\mathbf{v} \| = \| \mathcal{S}\mathbf{u} - \mathcal{S}\mathbf{v} \| \leq \| \mathcal{S} \| \| \mathbf{u} - \mathbf{v} \| < \| \mathbf{u} - \mathbf{v} \| .$$

Therefore, in controlling the operators $\mathcal{T}^{\mathrm{WJ}}$ and $\mathcal{T}^{\mathrm{SOR}}$ in the form of $\mathcal{T}^{\mathrm{WJ}}\mathbf{u} = \mathcal{S}^{\mathrm{WJ}}\mathbf{u} + \mathbf{c}^{\mathrm{WJ}}$ where

$$\mathcal{S}^{\mathrm{WJ}} = I - \omega D^{-1}A, \quad \mathbf{c}^{\mathrm{WJ}} = \omega D^{-1}\mathbf{G}$$

and $\mathcal{T}^{\mathrm{SOR}}\mathbf{u} = \mathcal{S}^{\mathrm{SOR}}\mathbf{u} + \mathbf{c}^{\mathrm{SOR}}$ where

$$\mathcal{S}^{\mathrm{SOR}} = I - \omega\,(D - \omega L)^{-1}\,A, \quad \mathbf{c}^{\mathrm{SOR}} = \omega\,(D - \omega L)^{-1}\,\mathbf{G}$$

be nonexpansive mapping, their weight parameter must be properly modified. The implemented of weight parameter ω for the operator $\mathcal{S}^{\mathrm{WJ}}$ and $\mathcal{S}^{\mathrm{SOR}}$ are defined as its norm less than one. Moreover, the optimal weight parameter ω_o in getting the smallest norm for each types of operator $\mathcal{S}$ are indicated on Table 3.3.3.

The different types of operator $\mathcal{S}$	Implement weight parameter ω	Optimal weight parameter ω_o
$\mathcal{S}^{\mathrm{WJ}}$	$0 < \omega < \dfrac{2}{\lambda_{\max}(D^{-1}A)}$	$\omega_o = \dfrac{2}{\lambda_{\min}(D^{-1}A) + \lambda_{\max}(D^{-1}A)}$
$\mathcal{S}^{\mathrm{SOR}}$	$0 < \omega < 2$	$\omega_o = \dfrac{2}{1+\sqrt{1-\rho^2}}$

Table 3.3.3: Implemented weight parameter and optimal weight parameter of operator $\mathcal{S}$.

The parameters $\lambda_{\max}(D^{-1}A)$ and $\lambda_{\min}(D^{-1}A)$ are the maximum and minimum eigenvalue of matrix $D^{-1}A$, respectively and ρ is the spectral radius of the iteration matrix of the Jacobi method.

Next, we introduce the proposed method in solving linear systems (3.3.30) by using every different two nonexpansive mapping $\mathcal{T}^i$ and $\mathcal{T}^j$. The generated sequence $\{\mathbf{u}^n\}$ is created iteratively by $\mathbf{u}^0 \in \mathbb{R}^n$ and

$$\begin{aligned} \mathbf{w}^{(n,s+1)} &= (1-\zeta_n)\mathbf{u}^{(n,s)} + \zeta_n \mathcal{T}^i \mathbf{u}^{(n,s)}, \\ \mathbf{z}^{(n,s+1)} &= (1-\vartheta_n)\mathcal{T}^j \mathbf{u}^{(n,s)} + \vartheta_n \mathcal{T}^i \mathbf{w}^{(n,s+1)}, \\ \mathbf{u}^{(n+1,s+1)} &= (1-\eta_n)\mathcal{T}^i \mathbf{z}^{(n,s+1)} + \eta_n \mathcal{T}^j \mathbf{w}^{(n,s+1)},\ n \geq 0, \end{aligned} \tag{3.3.32}$$

where the second superscript "s" denotes the number of iterations $s = 0, 1, \ldots, \widehat{S}_n$ and set

$$\zeta_n = \vartheta_n = \eta_n = 0.9, \tag{3.3.33}$$

as the default parameter. Since, we focus on the convergence of the proposed algorithm then the stability analysis in choosing the step sizes of time is not discussed in detail. The step size of time for the proposed algorithm are based on the smallest step size chosen from WJ and SOR method in solving linear system (3.3.30) generated from the discretization of the consideration problem

(3.3.26). In all computations, we used $\beta = 25$, $\Delta t = \Delta x^2/10$ (step size of time) and $\epsilon_d = 10^{-10}$. For testing purpose only, both computations are performed for $0 \le t \le 0.01$ (When $t \gg 0.05, u(x,t) \to 0$). The following stopping criterion is used

$$\|\mathbf{u}^{(n+1,\widehat{S}_n+1)} - \mathbf{u}^{(n+1,\widehat{S}_n)}\|_2 < \epsilon_d,$$

where $\widehat{S}_n$ denotes the number of the last iteration at time t^n and after that set $\mathbf{u}^{(n+1,\widehat{S}_n+1)} = \mathbf{u}^{(n,0)}$.

All computations are performed by using uniform grid of 101 nodes which corresponds to the solution of linear systems (3.3.30) with 99×99 sizes respectively. The weight parameter ω of the proposed algorithm set as its optimum weight parameter (ω_o) defined on Table 3.3.3. We apply the WJ, SOR, GS and the proposed algorithm with two different operators $\mathcal{T}^i$ and $\mathcal{T}^j$ (Proposed algorithm with $\mathcal{T}^i - \mathcal{T}^j$) in getting the solution of linear system (3.3.30) of heat problem with Dirichlet boundary conditions and initial data (3.3.26). For the proposed algorithm, the nonexpansive mapping $\mathcal{T}^i$ are chosen from the following operators: $\mathcal{T}^{\mathrm{GS}}, \mathcal{T}^{\mathrm{WJ}}$ and $\mathcal{T}^{\mathrm{SOR}}$.

Let consider the following heat problem:

$$\begin{aligned} &u_t = \beta u_{xx} + 0.4\beta(4\pi^2 - 1)e^{-4\beta t}\cos(4\pi x), \quad 0 \le x \le 1, \quad 0 < t < t_s, \\ &u(x,0) = \cos(4\pi x)/10, \quad u(0,t) = e^{-4\beta t}/10, \quad u(1,t) = e^{-4\beta t}/10, \\ &\qquad u(x,t) = e^{-4\beta t}\cos(4\pi x)/10. \end{aligned} \tag{3.3.34}$$

The results of the basic iterative methods (WJ, GS, SOR) and the proposed algorithms are demonstrated and discussed on the following cases:

Case I: WJ Method,

Case II: GS Method,

Case III: SOR Method,

Case IV: The proposed algorithm with $\mathcal{T}^{\mathrm{WJ}}$–$\mathcal{T}^{\mathrm{GS}}$,

Case V: The proposed algorithm with $\mathcal{T}^{\mathrm{WJ}}$–$\mathcal{T}^{\mathrm{SOR}}$,

Case VI: The proposed algorithm with $\mathcal{T}^{\mathrm{GS}}$–$\mathcal{T}^{\mathrm{WJ}}$,

Case VII: The proposed algorithm with $\mathcal{T}^{\mathrm{GS}}$–$\mathcal{T}^{\mathrm{SOR}}$,

Case VIII: The proposed algorithm with $\mathcal{T}^{\mathrm{SOR}}$–$\mathcal{T}^{\mathrm{WJ}}$,

Case IX: The proposed algorithm with $\mathcal{T}^{\mathrm{SOR}}$–$\mathcal{T}^{\mathrm{GS}}$.

The exact error are measured by using $\|\mathbf{u}^{\mathbf{n}} - \mathbf{u}\|_2/\|\mathbf{u}\|_2$. Figure 3.3.1 shows the approximate solution of problem (3.3.34) with 101 nodes at $t = 0.01$ by using the basic iterative methods and the proposed algorithm with cases I - IX.

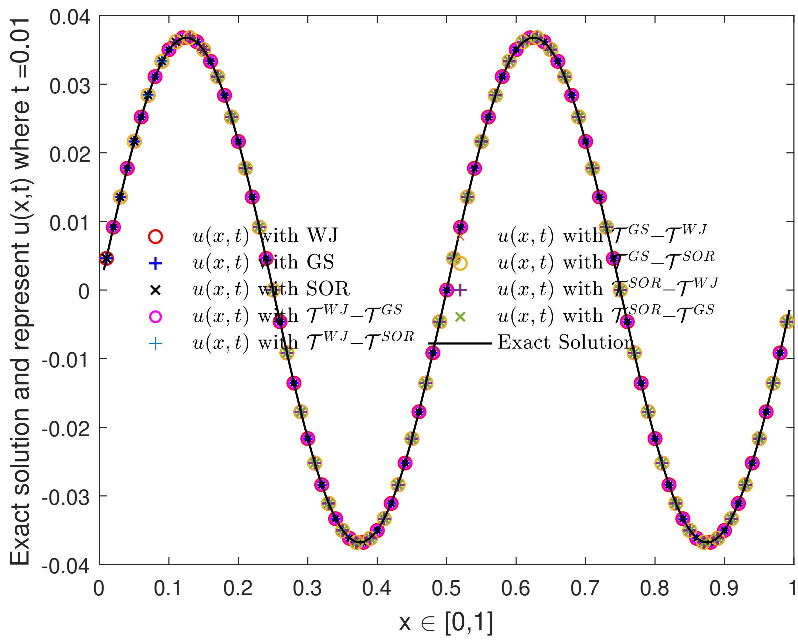

Figure 3.3.1: Approximate solutions of the basic iterative methods and all cases of the proposed algorithms to problem (3.3.34) with 101 nodes.

It can be seen from Figure 3.3.1 that all numerical solution matches the analytical solution reasonably well. Thus, It can be concluded that all sequences generated by the proposed method with two different operators $\mathcal{T}^i$ and $\mathcal{T}^j$ converge to their common fixed point solution.

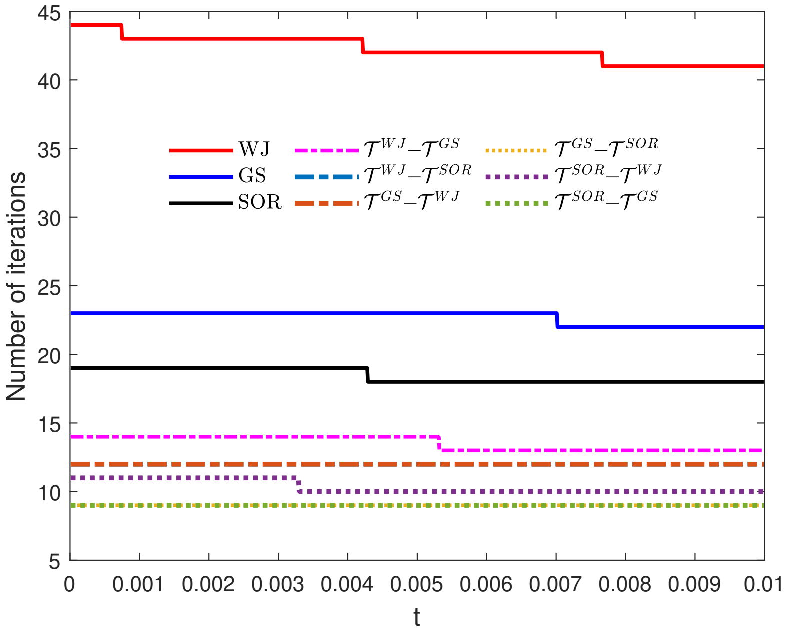

Figure 3.3.2: The evolution of iterations number for GS, WJ, SOR and the proposed algorithm to problem (3.3.26) with $\beta = 25$ and $t \in (0, 1]$.

Figure 3.3.2 shows the trend of the iterations number for the basic iterative methods and the proposed algorithm with Case IV - Case IX in solving problem (3.3.30) generates from the discretization of the consideration problem (3.3.34) with 101 nodes. It can be seen from Figure 3.3.2 that the proposed method with two different operators $\mathcal{T}^i$ and $\mathcal{T}^j$ uses less number of iteration on each step of time compare with he basic iterative methods. And the proposed methods with $\mathcal{T}^{\mathrm{GS}}$–$\mathcal{T}^{\mathrm{SOR}}$ and $\mathcal{T}^{\mathrm{SOR}}$–$\mathcal{T}^{\mathrm{GS}}$ give us the smallest number of iteration on each step of time.

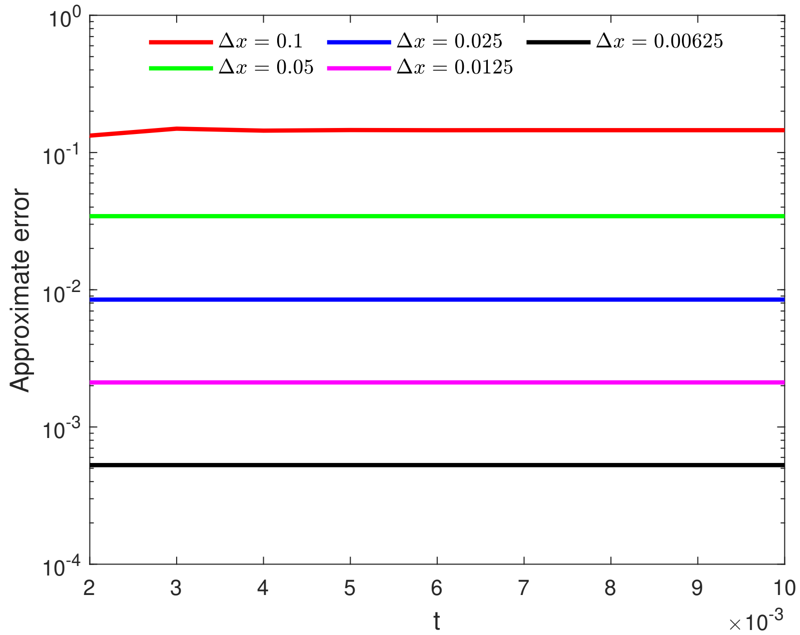

Figure 3.3.3: The evolution of iterations number for the basic iterative methods and the proposed algorithm to problem (3.3.26) with 101 nodes and $t \in (0, 1]$.

Next, the proposed algorithm with $\mathcal{T}^{\text{SOR}}$–$\mathcal{T}^{\text{GS}}$ is chosen to test and verify the order of accuracy for the presented FDS in solving heat equation (3.3.34). And, all computations are performed by using uniform grids of $11, 21, 41, 81$ and 161 nodes that correspond to the solution of the discretization of heat equation problem (3.3.34) with $\Delta x = 0.1, 0.05, 0.025, 0.0125, 0.0625$, respectively. We found that the proposed algorithm with $\mathcal{T}^{\text{SOR}}$–$\mathcal{T}^{\text{GS}}$ are seen to be second order of accuracy (See on Figure 3.3.3) when the distance between the graphs of all computational grid sizes are compared. That is the order of accuracy of the proposed algorithm with $\mathcal{T}^{\text{SOR}}$–$\mathcal{T}^{\text{GS}}$ agrees with the construction of their FDS. Figure 3.3.4 shows the trend of the average iterations number for the basic iterative methods compare with all cases of the proposed algorithms in solving the discretization of consideration problem (3.3.34) with various grid sizes.

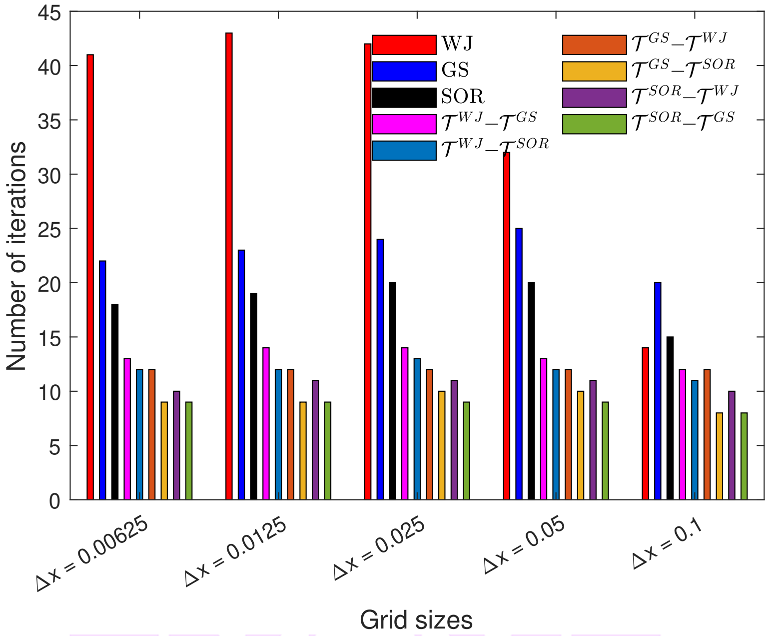

Figure 3.3.4: The evolution of the average iterations number on each step of time for GS, WJ, SOR and the proposed algorithm to problem (3.3.26)) with $\vartheta = 25$ and $t \in (0, 1]$.

It can be seen from this figure that the average number of iteration on each step of time for the proposed method with cases IV - IX are less than the average number of iteration on each step of time for WJ, GS and SOR methods. Moreover, the average number of iteration on each step of time for the proposed algorithm with $\mathcal{T}^{\text{SOR}} - \mathcal{T}^{\text{GS}}$ give us the smallest number of iterations for all of the consideration grid sizes. However, even if using a small amount of iteration per step of time show the excellent performance of the proposed method but the stability condition of the proposed algorithm need to be considered carefully as chosen for the results of the stability analysis with time.

3.3.4.2 Image deblurring problems

Let B is the degraded image of the true image U in the matrix form of $\tilde{m}$ rows and $\tilde{n}$ columns $(B, U \in \mathbb{R}^{\tilde{m}\times\tilde{n}})$. The key to obtaining the image restoration model is to rearrange the elements of the images B and U into the column vectors by stacking the columns of these images into two long vectors $\mathbf{b}$ and $\mathbf{u}$ where $\mathbf{b} = \mathrm{vec}(B)$ and $\mathbf{u} = \mathrm{vec}(U)$, both of length $n = \tilde{m}\tilde{n}$. The image restoration problem can be modelled in one dimensional vector by the following linear equation system:

$$\mathbf{b} = M\mathbf{u}, \tag{3.3.35}$$

where $\mathbf{u} \in \mathbb{R}^n$ is an original image, $\mathbf{b} \in \mathbb{R}^n$ is the observed image, $M \in \mathbb{R}^{n\times n}$ is the blurring operation and $n = \tilde{m}\tilde{n}$. In order to solve problem (3.3.35), we aim to approximate the original image, vector $\mathbf{b}$, which is known as the following least squares (LS) problem:

$$\min_{\mathbf{u}} \frac{1}{2}\|\mathbf{b} - M\mathbf{u}\|_2^2, \tag{3.3.36}$$

where $\|.\|_2$ is defined by $\|\mathbf{u}\|_2 = \sqrt{\sum_{i=1}^{n} |u_i|^2}$.

Next, we will apply our method for solving the LS problem (3.3.36) and the image restoration problems (3.3.35) by setting as follows: Let $M \in \mathbb{R}^{n\times n}$ be a degraded matrix and $\mathbf{b} \in \mathbb{R}^n$. We obtain the following proposed methods to find the common solution of the image restoration problem

$$\begin{aligned} \mathbf{w}_n &= (1-\zeta_n)\mathbf{u}_n + \zeta_n \mathcal{T}\mathbf{u}_n, \\ \mathbf{z}_n &= (1-\vartheta_n)\mathcal{T}\mathbf{u}_n + \vartheta_n \mathcal{T}\mathbf{w}_n, \\ \mathbf{u_{n+1}} &= (1-\eta_n)\mathcal{T}\mathbf{z}_n + \eta_n \mathcal{T}\mathbf{w}_n, \end{aligned} \tag{3.3.37}$$

where $\mathcal{T}\mathbf{u} = \mathbf{u} - \mu M^T(M\mathbf{u} - \mathbf{b})$, $\mu \subset \big(0, 2/||M^TM||_2\big)$ and $\eta_n, \vartheta_n, \zeta_n \in (0,1)$ for all $n \in \mathbb{N}$.

The goal on image restoration problem is to find the original image from the observed image without knowing which one is the blurring matrix. However, the blurring matrix M must be known in applying algorithm (3.3.37). Now, we present the new idea in solving the image restoration problem when the observed image $\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_N$ can be restored by using the blurring matrices $M_1, M_2, ..., M_N$ with the different quality respectively ($\mathbf{b}_i = M_i\mathbf{u}, i = 1, 2, \dots, N$). Let us consider the following problems:

$$\min_{\mathbf{u}\in\mathbb{R}^n} \frac{1}{2}\|M_1\mathbf{u} - \mathbf{b}_1\|_2^2, \min_{\mathbf{u}\in\mathbb{R}^n} \frac{1}{2}\|M_2\mathbf{u} - \mathbf{b}_2\|_2^2, ..., \min_{\mathbf{u}\in\mathbb{R}^n} \frac{1}{2}\|M_N\mathbf{u} - \mathbf{b}_N\|_2^2, \qquad (3.3.38)$$

where $\mathbf{u}$ is the originally true image (common solution), M_i is the blurred matrix, $\mathbf{b}_i$ is the blurred image by the blurred matrix M_i. However, there are many degraded matrices must be known. For simplicity, we give an example in applying algorithm (3.3.37) in finding the original image $\mathbf{u}$ with a pair of blurring matrices M_i and M_j:

$$\begin{aligned} \mathbf{w}^n &= (1-\zeta_n)\mathbf{u}^n + \zeta_n\mathcal{T}^i\mathbf{u}^n, \\ \mathbf{z}^n &= (1-\vartheta_n)\mathcal{T}^j\mathbf{u}^n + \vartheta_n\mathcal{T}^i\mathbf{w}^n, \\ \mathbf{u}^n &= (1-\eta_n)\mathcal{T}^i\mathbf{z}^n + \eta_n\mathcal{T}^j\mathbf{w}^n, \end{aligned} \qquad (3.3.39)$$

where $\mathcal{T}^k\mathbf{u} = \mathbf{u} - \mu_k M_k^T(M_k\mathbf{u} - \mathbf{b})$. The implemented algorithm (3.3.39) is proposed in solving the image restoration problem by using a pair of the blurring matrices M_i and M_j with default parameter (3.3.33) and $\mu_k = 1/\|M_k^T M_k\|_2$ and called it as the proposed algorithm with $\mathcal{T}^i$–$\mathcal{T}^j$.

In case of $\mathcal{T}^i = \mathcal{T}^j$, we called it as the proposed algorithm with $\mathcal{T}^i$.

Figure 3.3.5: The original RGB image with matrix size $404 \times 438 \times 3$.

The original RGB format shown in Figure 3.3.5 is used to demonstrate the practicability of the proposed algorithm. The relative Cauchy error and the relative image error are measured by using max-norm $\|\mathbf{u}_n - \mathbf{u}_{n-1}\|_\infty / \|\mathbf{u}\|_\infty$ and $\|\mathbf{u}_n - \mathbf{u}\|_\infty / \|\mathbf{u}\|_\infty$, respectively. The performance of the comparing algorithms at $\mathbf{u}_n$ on image deblurring process is measured quantitatively by the means of the peak signal-to-noise ratio (PSNR), which is defined by

$$\text{PSNR}(\mathbf{u}_n) = 20\log_{10}\Big(\frac{255^2}{MSE}\Big),$$

where MSE $= \|\mathbf{u}_n - \mathbf{u}\|_2^2$. Three different types of the original RGB image degraded by the blurring matrices M_1, M_2 and M_3 are shown in Figure 3.3.6. These are used to test the implemented algorithm.

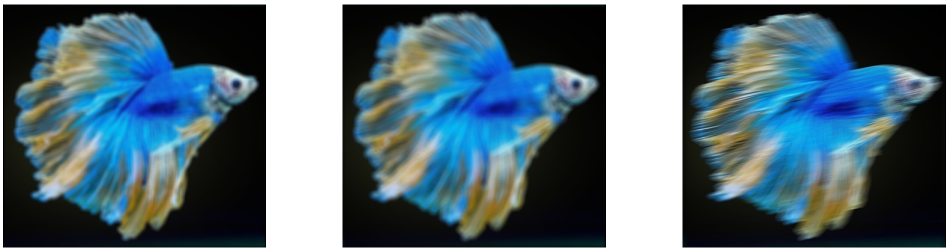

Figure 3.3.6: The original RGB image degraded by blurred matrices M_1, M_2 and M_3 respectively.

Next, we present restoration of images that have been corrupted by the following blur types:

Type I Gaussian blur of filter size 9×9 with standard deviation $\sigma = 4$ (The original image has been degraded by the blurring matrix M_1).

Type II Out of focus blur (Disk) with radius $r = 6$ (The original image has been degraded by the blurring matrix M_2).

Type III Motion blur specifying with motion length of 21 pixels (len $= 21$) and motion orientation 11° ($\theta = 11$) (The original image has been degraded by the blurring matrix M_3).

Since, the using image U and three different types of the blurring image B (see on Figure 3) are represented in the red-green-blue component. Then, we denote U_r, U_g, U_b and B_r, B_g, B_b as the gray-scale images that constitute the red-green-blue channels of the using image U and the blurring image B respectively. Thus, we define the column vector $\mathbf{u}$ and $\mathbf{b}$ from color image U and B and both of length $n = 3\tilde{m}\tilde{n}$. After that, we apply the proposed algorithms in getting the common solution of the image restoration problem with these three blurring matrices. Both theoretical and experimental results demonstrate the convergence properties of the proposed algorithm with the permutation of the blurring matrices M_1, M_2 and M_3 are demonstrated and discussed on the following cases:

Case I: Algorithm 3.3.39 with $\mathcal{T}^1$, **Case II:** Algorithm 3.3.39 with $\mathcal{T}^2$,

Case III: Algorithm 3.3.39 with $\mathcal{T}^3$, **Case IV:** Algorithm 3.3.39 with $\mathcal{T}^1$–$\mathcal{T}^2$,

Case V: Algorithm 3.3.39 with $\mathcal{T}^1$–$\mathcal{T}^3$, **Case VI:** Algorithm 3.3.39 with $\mathcal{T}^2$–$\mathcal{T}^1$,

Case VII: Algorithm 3.3.39 with $\mathcal{T}^2$–$\mathcal{T}^3$, **Case VIII:** Algorithm 3.3.39 with $\mathcal{T}^3$–$\mathcal{T}^1$,

Case IX: Algorithm 3.3.39 with $\mathcal{T}^3$–$\mathcal{T}^2$.

Figure 3.3.7 shows the plots behavior of the relative Cauchy and image error of the reconstructed RGB image in all cases.

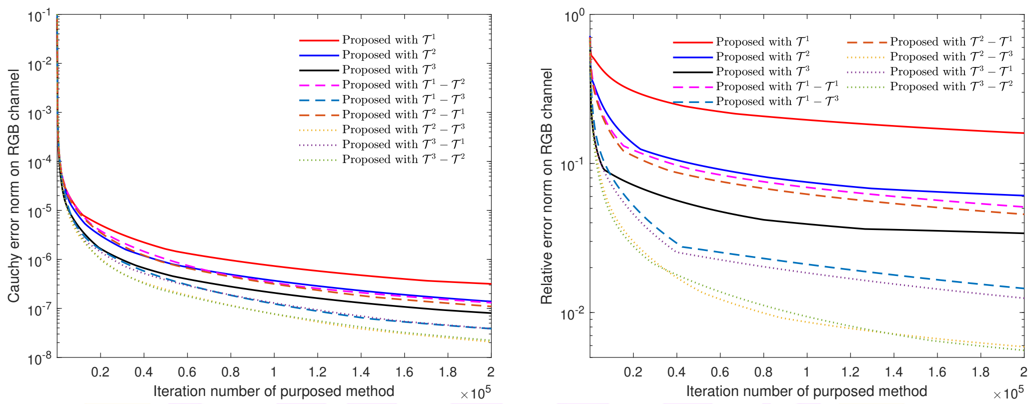

Figure 3.3.7: The relative Cauchy and image error plots of the proposed algorithm with cases I - IX.

The relative Cauchy and image error plots guarantee that the presented method converge to the common solution of restoration problem (3.3.39) in all cases. These plots show the validity and confirm the convergence of the proposed methods. Figure 3.3.8 shows the all cases of the PSNR plot for the proposed methods with $200,000^{\text{th}}$ iterations. Based on the PSNR plots on Figure 3.3.8, all restored image using the proposed algorithms in solving the deblurring problem get the quality improvements when the iteration number increases. Moreover, the PSNR quality of the observed image is improved when the proposed method with a pair of $\mathcal{T}^i$–$\mathcal{T}^j$ and $i \neq j$ is used for solving deblurring problem compare with the proposed method in which the only one blurring matrix is used. And, the best case in recovering the observed image occurs when the proposed methods with $\mathcal{T}^2$–$\mathcal{T}^3$ and $\mathcal{T}^3$–$\mathcal{T}^2$ are used.

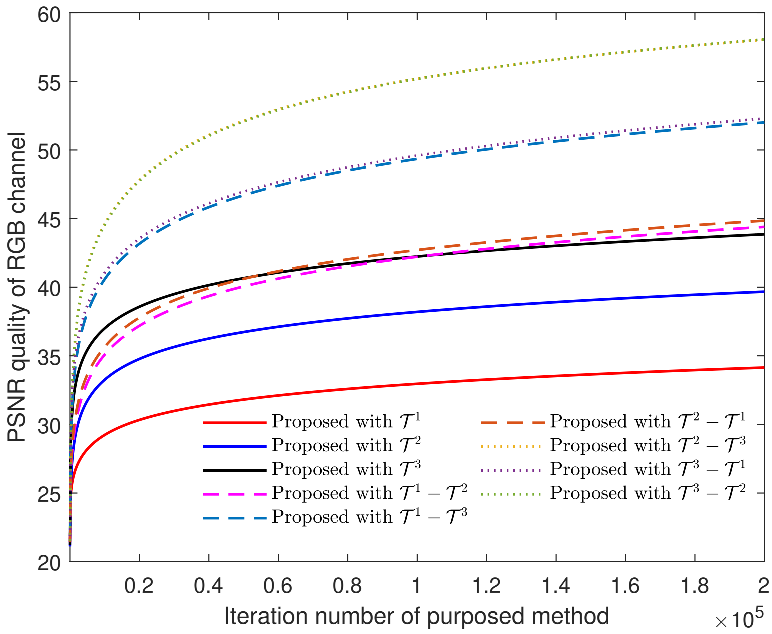

Figure 3.3.8: The comparison plots of the PSNR quality of the proposed algorithm with cases I - IX.

Figure 3.3.9 demonstrates the crop of reconstructed RGB image presented in 500^{th} iteration by using the proposed algorithms in getting the common solution of the restoration problem with operators $\mathcal{T}^1$ and $\mathcal{T}^2$, $\mathcal{T}^3$ and $\mathcal{T}^3$–$\mathcal{T}^2$ respectively. It can be seen from these figures that the quality of restored image by using the proposed algorithms with $\mathcal{T}^3$–$\mathcal{T}^2$ get the smooth quality of the using degraded image.

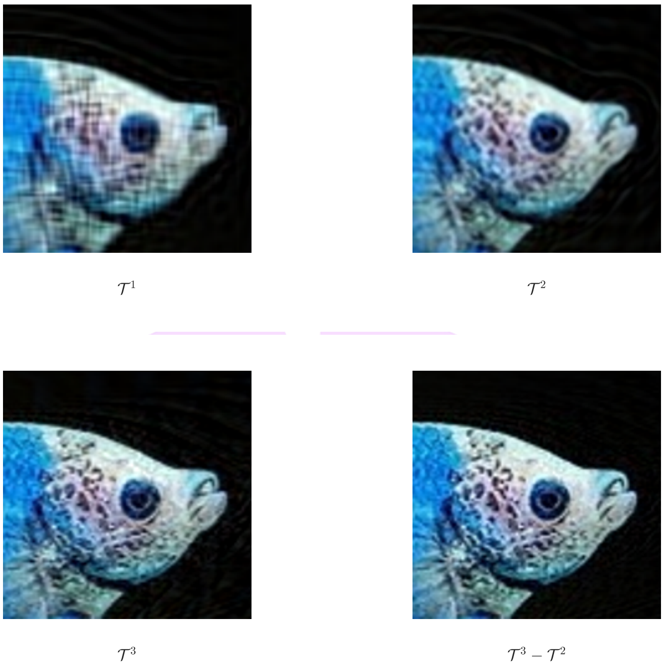

Figure 3.3.9: The reconstructed images being used the proposed algorithms with operators $\mathcal{T}^1$, $\mathcal{T}^2$, $\mathcal{T}^3$ and $\mathcal{T}^3{-}\mathcal{T}^2$, respectively present in 500^{th} iterations

Next, we will compare the effective of the proposed methods with Mann iteration [34], Ishikawa iteration [24], S-iteration [2], Noor iteration [35], AK-iteration [64], Sahu et al. iteration [53] and SP-iteration [41] through the PSNR plots on Figure 3.3.10. And, the parameters ζ_n, ϑ_n and η_n of the comparative algorithms are set with the default parameter (3.3.33). Since all comparing methods use only one blurring matrix on their algorithms, the proposed method with cases I - III is used to compare our methods' effectiveness. The PSNR plots of the proposed method with case VII have been presented and demonstrated the effectiveness of the proposed methods.

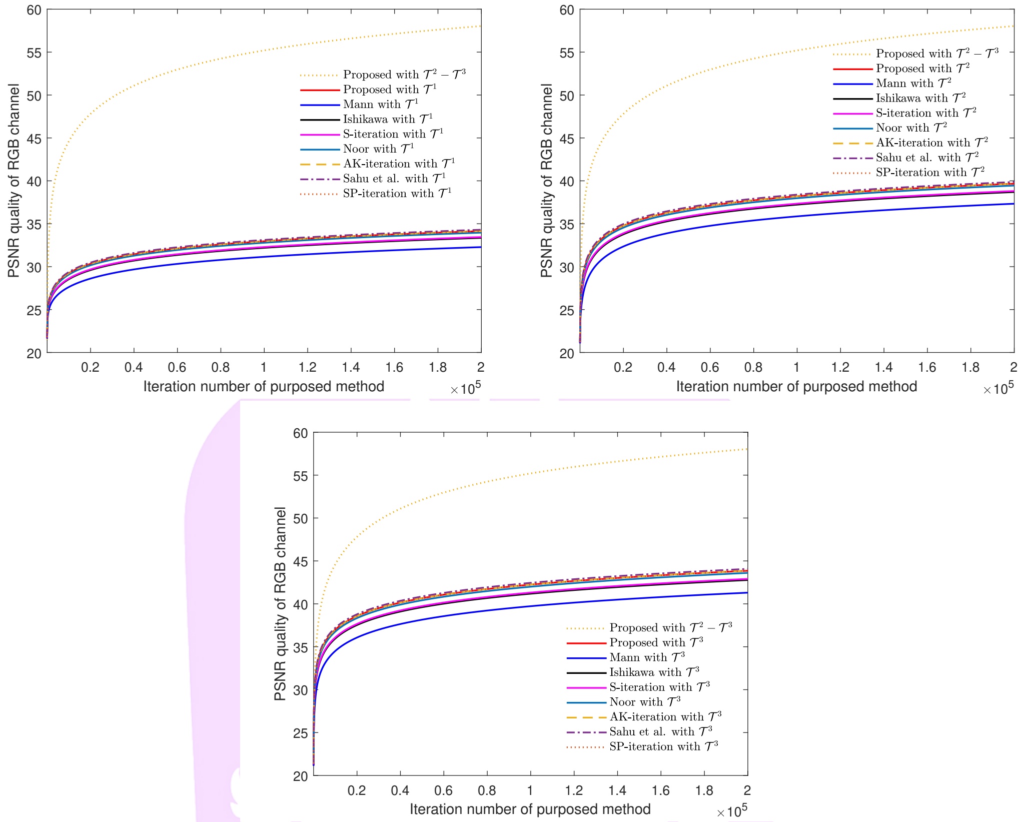

Figure 3.3.10: The PSNR plots for all comparing methods comparing the proposed method presented in $100,000^{\text{th}}$ iterations.

It can be seen from Figure 3.3.10 that the PSNR quality of AK, Sahu, et al., and SP iterations are better than the proposed methods. However, the proposed method has been designed to be used with a pair of $\mathcal{T}^j$–$\mathcal{T}^k$ where $j \neq k$. With this advantage, we found that the proposed method with cases IV - IX is more effective than AK, Sahu, et al., and SP iterations (see Figure 3.3.8). And, when we compare the results of Noor, AK, Sahu, et al., and SP iterations with case VII of the proposed method, it is clearly seen that the proposed way is much more effective.

3.3.4.3 Signal recovering problems

In signal processing, compressed sensing can be modeled as the following under determinated linear equation system

$$\mathbf{y} = A\mathbf{u} + \nu,$$

where $\mathbf{u} \in \mathbb{R}^n$ is an original signal with n components to be recovered, $\nu, \mathbf{y} \in \mathbb{R}^m$ are noise and the observed signal with noisy for m components respectively and $A \in \mathbb{R}^{m\times n}$ is a degraded matrix. Finding the solutions of previous determinated linear equation system can be seen as solving the LASSO problem

$$\min_{\mathbf{u}\in\mathbb{R}^N} \frac{1}{2}\|\mathbf{y} - A\mathbf{u}\|_2^2 + \lambda\|\mathbf{u}\|_1, \tag{3.3.40}$$

where $\lambda > 0$. As a result various techniques and iterative schemes have been developed to solve the LASSO problem. We can apply our method for solving the LASSO problem (3.3.40) by setting $\mathcal{T}\mathbf{u} = prox_{\mu g}(\mathbf{u} - \mu\nabla f(\mathbf{u}))$ where $f(\mathbf{u}) = \|\mathbf{y} - A\mathbf{u}\|_2^2/2$, $g(\mathbf{u}) = \lambda\|\mathbf{u}\|_1$ and $\nabla f(\mathbf{u}) = A^T(A\mathbf{u} - \mathbf{y})$.

Next, we give an example in applying our algorithm in signal recovery problems. Let $A \in \mathbb{R}^{m\times n}(m < n)$ be a degraded matrix and $\mathbf{y}, \nu \in \mathbb{R}^m$, we obtain the following proposed methods to find the solution of the signal recovery problem:

$$\begin{aligned} \mathbf{w}_n &= (1-\zeta_n)\mathbf{u}_n + \zeta_n\mathcal{T}\mathbf{u}_n, \\ \mathbf{z}_n &= (1-\vartheta_n)\mathcal{T}\mathbf{u}_n + \vartheta_n\mathcal{T}\mathbf{w}_n, \\ \mathbf{u_{n+1}} &= (1-\eta_n)\mathcal{T}\mathbf{z}_n + \eta_n\mathcal{T}\mathbf{w}_n, \end{aligned} \tag{3.3.41}$$

where $\mathcal{T}\mathbf{u} = prox_{\mu g}\left(\mathbf{u} - \mu A^T\left(A\mathbf{u} - \mathbf{y}\right)\right)$, $\mu \in \left(0, 2/||A^tA||_2\right)$ and $\eta_n, \vartheta_n, \zeta_n \in (0,1)$ for all $n \in \mathbb{N}$. The goal in signal recovery problem is to find the original signal from the observed signal without knowing of the degraded signal operator A and noise ν. However, the degraded signal operator A must be known in applying algorithm (3.3.41). Now, we present the method in solving the signal recovery problem when the observed signal $\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_N$ can be recovered by using the known degraded matrices $A_1, A_2, ..., A_N$ ($\mathbf{y}_i = A_i\mathbf{u} + \nu_i, \nu_i = \lambda_i\|u\|_1, i = 1, 2, \ldots, N$). Let us consider the following problems:

$$\begin{aligned} \min_{\mathbf{u}\in\mathbb{R}^N} \frac{1}{2}\|A_1\mathbf{u} - \mathbf{y}_1\|_2^2 &+ \lambda_1\|\mathbf{u}\|_1, \\ \min_{\mathbf{u}\in\mathbb{R}^N} \frac{1}{2}\|A_2\mathbf{u} - \mathbf{y}_2\|_2^2 &+ \lambda_2\|\mathbf{u}\|_1, \\ &\vdots \\ \min_{\mathbf{u}\in\mathbb{R}^N} \frac{1}{2}\|A_N\mathbf{u} - \mathbf{y}_N\|_2^2 &+ \lambda_N\|\mathbf{u}\|_1, \end{aligned} \tag{3.3.42}$$

where a true signal $\mathbf{u}$ is common solution of problem (3.3.42). That is, we will find the true signal $\mathbf{u}$ through the common solution of N LASSO problems on problem (3.3.42). However, there are many degraded signal operators must be known. For simplicity, we give an example in applying our algorithm in finding the common solution $\mathbf{u}$ for a pair of LASSO problem on equation (3.3.42):

$$\begin{aligned} \mathbf{w}_n &= (1-\zeta_n)\mathbf{u}_n + \zeta_n\mathcal{T}_j\mathbf{u}_n, \\ \mathbf{z}_n &= (1-\vartheta_n)\mathcal{T}_k\mathbf{u}_n + \vartheta_n\mathcal{T}_j\mathbf{w}_n, \\ \mathbf{u_{n+1}} &= (1-\eta_n)\mathcal{T}_j\mathbf{z}_n + \eta_n\mathcal{T}_k\mathbf{w}_n, \end{aligned} \tag{3.3.43}$$

where

$$\mathcal{T}_i\mathbf{u} = prox_{\mu_i g_i}\left(\mathbf{u} - \mu_i A_i^T\left(A_i\mathbf{u} - \mathbf{y}_i\right)\right),$$

$$g_i(\mathbf{u}) = \lambda_i\|\mathbf{u}\|_1$$

and

$$\mu_i = 1/\|A_i^T A_i\|_2,$$

$i = 1, 2, ..., N$.

The implemented algorithm (3.3.43) is proposed in solving the signal recovery problem by using a pair of the degraded signal perators A_j and A_k and set $\zeta_n = \vartheta_n = \eta_n = 0.9$ and called it as the proposed algorithm with operator $\mathcal{T}_j - \mathcal{T}_k$. In case of $\mathcal{T}_j = \mathcal{T}_k$, we called it as the proposed algorithm with $\mathcal{T}_j$.

Next, some experiments are provided to illustrate the convergence and the effectiveness of the proposed algorithm (3.3.43). The original signal $\mathbf{u}$ with $n = 1024$ generated by the uniform distribution in the interval $[-2, 2]$ with 70 nonzero elements is used to create the observation signal $\mathbf{y}_i = A_i\mathbf{u} + \nu_i, i = 1, 2, 3$ with $m = 512$ (see on Figure 3.3.11).

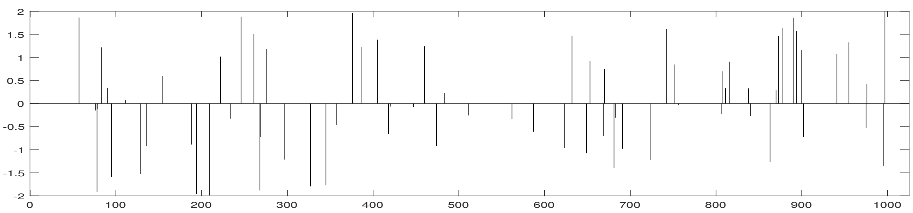

Figure 3.3.11: Original Signal ($\mathbf{u}$) with 70 nonzero elements.

The process is started when the signal initial data $\mathbf{u}_0$ with $n = 1024$ is picked randomly (see on Figure 3.3.12).

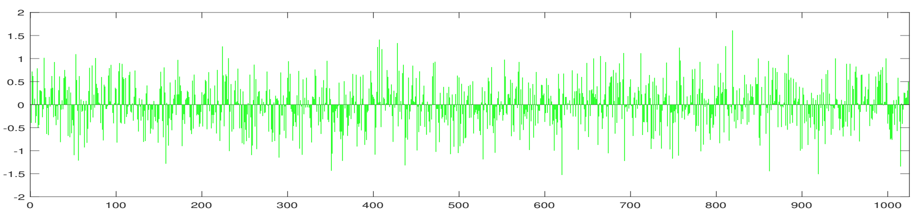

Figure 3.3.12: Initial Signals $\mathbf{u}_0$.

The observation signal $\mathbf{y}_i$ shows on Figure 3.3.13.

Figure 3.3.13: Degraded Signals $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ respectively.

The matrices A_i that generated by the normal distribution with mean zero and variance one and the white Gaussian noise $\nu_i, i = 1, 2, 3$ (see on Figure 3.3.14).

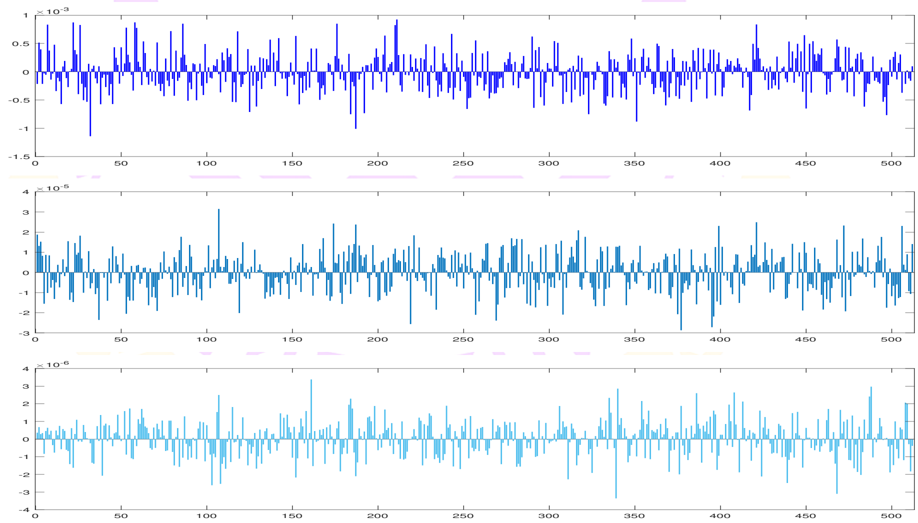

Figure 3.3.14: Noise Signals ν_1, ν_2, and ν_3 respectively.

Both theoretical and experimental results demonstrate the convergence properties of the proposed algorithm with the permutation of the blurring matrices A_1, A_2 and A_3 are demonstrated and discussed on the following cases:

Case I: Algorithm 3.3.43 with $\mathcal{T}_1$, **Case II:** Algorithm 3.3.43 with $\mathcal{T}_2$,

Case III: Algorithm 3.3.43 with $\mathcal{T}_3$, **Case IV:** Algorithm 3.3.43 with $\mathcal{T}_1$–$\mathcal{T}_2$,

Case V: Algorithm 3.3.43 with $\mathcal{T}_1$–$\mathcal{T}_3$, **Case VI:** Algorithm 3.3.43 with $\mathcal{T}_2$–$\mathcal{T}_1$,

Case VII: Algorithm 3.3.43 with $\mathcal{T}_2$–$\mathcal{T}_3$, **Case VIII:** Algorithm 3.3.43 with $\mathcal{T}_3$–$\mathcal{T}_1$,

Case IX: Algorithm 3.3.43 with $\mathcal{T}_3$–$\mathcal{T}_2$.

The relative Cauchy error and the relative error are measured by using second-norm $\|\mathbf{u}_n - \mathbf{u}_{n-1}\|_2/\|\mathbf{u}\|_2$ and $\|\mathbf{u}_n - \mathbf{u}\|_2/\|\mathbf{u}\|_2$, respectively. The performance of the proposed method at n^{th} iteration is measured quantitatively by the means of the the signal-to-noise ratio (SNR), which is defined by

$$\mathrm{SNR}(\mathbf{u}_n) = 20\log_{10}\left(\frac{\|\mathbf{u}_n\|_2}{\|\mathbf{u}_n - \mathbf{u}\|_2}\right),$$

where $\mathbf{u}_n$ is the recovered signal at n^{th} iteration by using the proposed method. The relative Cauchy error, relative signal error and SNR quality of the proposed methods for recovering the degraded signal are shown on Figure 3.3.15.

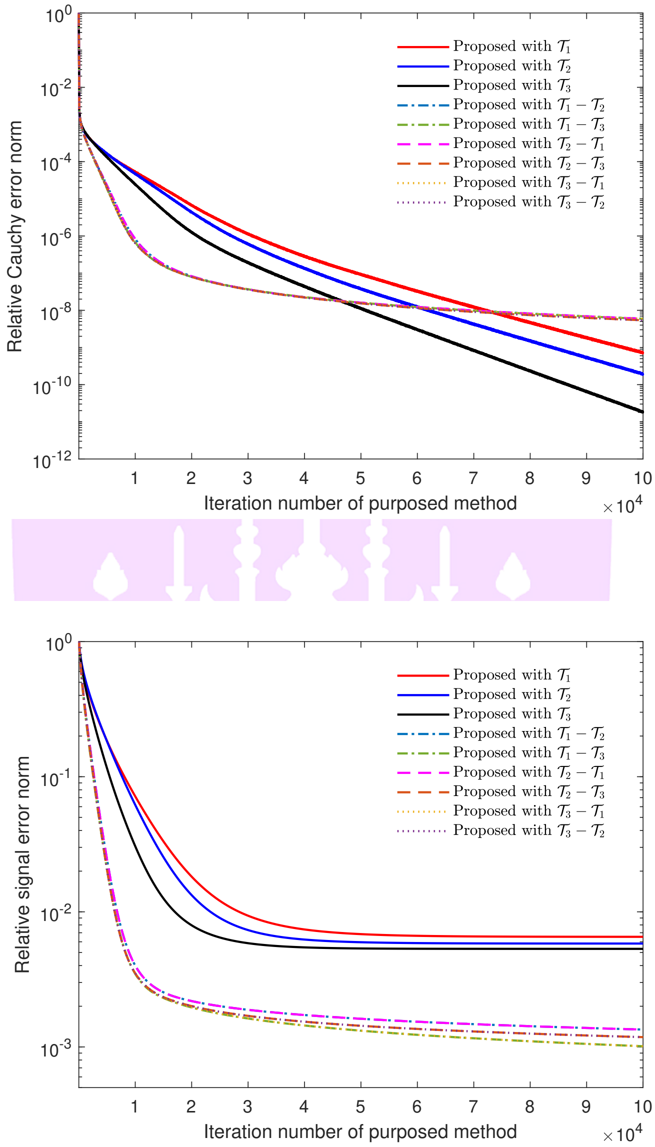

Figure 3.3.15: The relative Cauchy and signal error plots of the proposed methods with cases I - IX in recovering the observed signal.

The relative error of Cauchy plot guarantee that all cases of the presented method converge to common solution of the signal recovery problem (3.3.42) with $N = 2$. It is remarkable that, the relative signal error plot decreases until it converge to some constant value. Figure 3.3.16 shows the all cases of the SNR plot for the proposed methods with $100,000^{\text{th}}$ iterations. For the SNR quality plot on Figure 3.3.16, it can be seen that the SNR value increases until it converges to some constant value. Through these results, it can be concluded that the solution of the signal recovery problem solved by the proposed algorithm get the quality improvements of the observed signal. Moreover, the SNR quality of the observed signal has been greatly improved when the proposed method with $\mathcal{T}_j - \mathcal{T}_k$ and $\mathcal{T}_j \neq \mathcal{T}_k$ is used for solving the signal recovery problem. And, the optimal case in recovering the observed signal occurred when the proposed method with $\mathcal{T}_1 - \mathcal{T}_3$ is used.

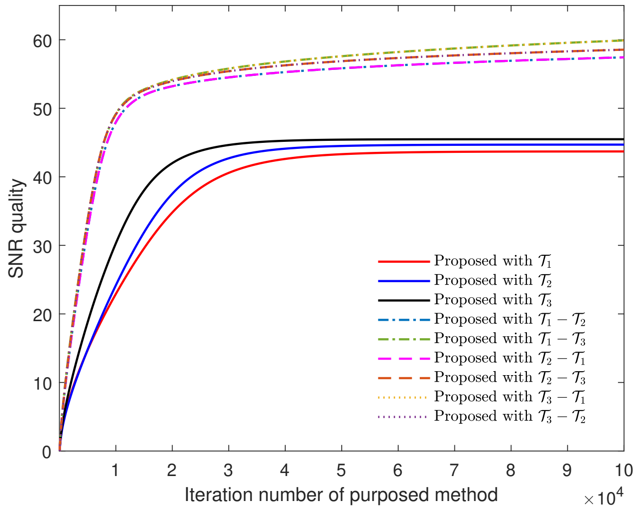

Figure 3.3.16: The comparison plots of the SNR quality of the proposed algorithm with cases I - IX in recovering the observed signal.

Figure 3.3.17 shows the restored signal by using one of the optimal proposed algorithm with operators $\mathcal{T}_1 - \mathcal{T}_3$ compare with proposed algorithms with operators $\mathcal{T}_1$, $\mathcal{T}_2$ and $\mathcal{T}_3$.

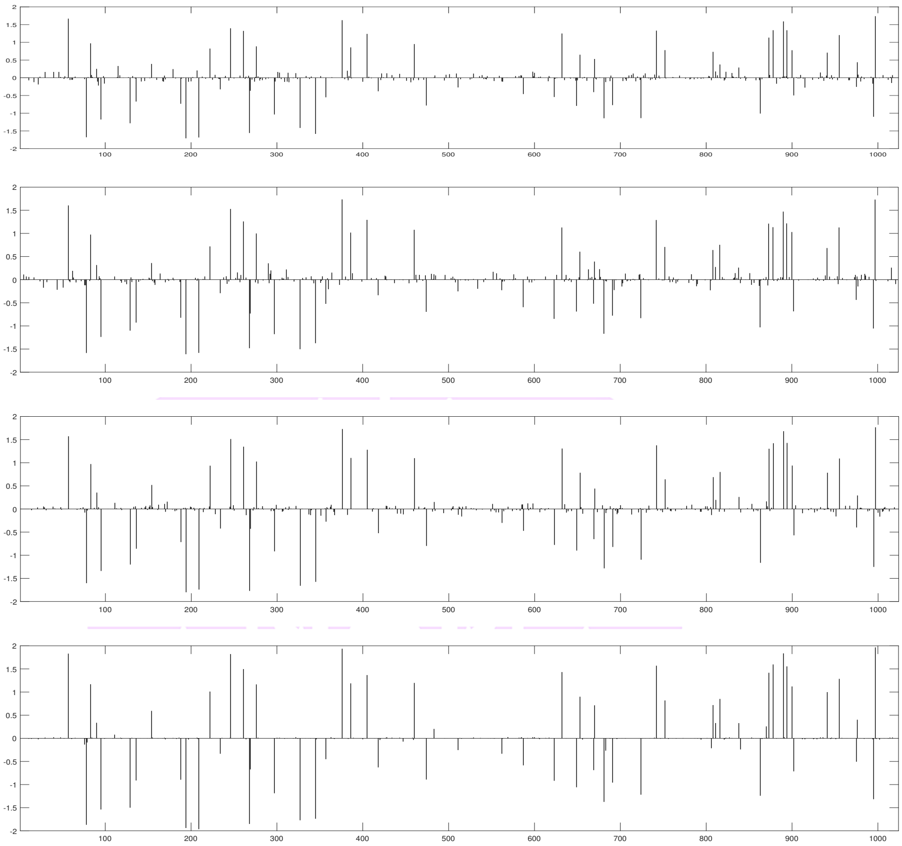

Figure 3.3.17: Recovering signals being used the proposed algorithms with operators $\mathcal{T}_1$ and $\mathcal{T}_2$, $\mathcal{T}_3$ and $\mathcal{T}_1$–$\mathcal{T}_3$, respectively presented in $4,000^{\text{th}}$ iterations.

Next, we will compare the effectiveness of the proposed methods with Mann [34], Ishikawa [24], S–iteration [2], Noor iteration [35], AK-iteration [64], Sahu et al. iteration [53] and SP–iteration [41] through the PSNR plots on Figure 3.3.18. And, the parameters ζ_n, ϑ_n and η_n of the comparative algorithms are set with the default parameter (3.3.33). Since all comparing methods use only one blurring matrix on their algorithms, the proposed method with cases I - III is used to compare our methods effectiveness. The SNR plots of the proposed method with case V have been presented and demonstrated the effectiveness of the proposed methods.

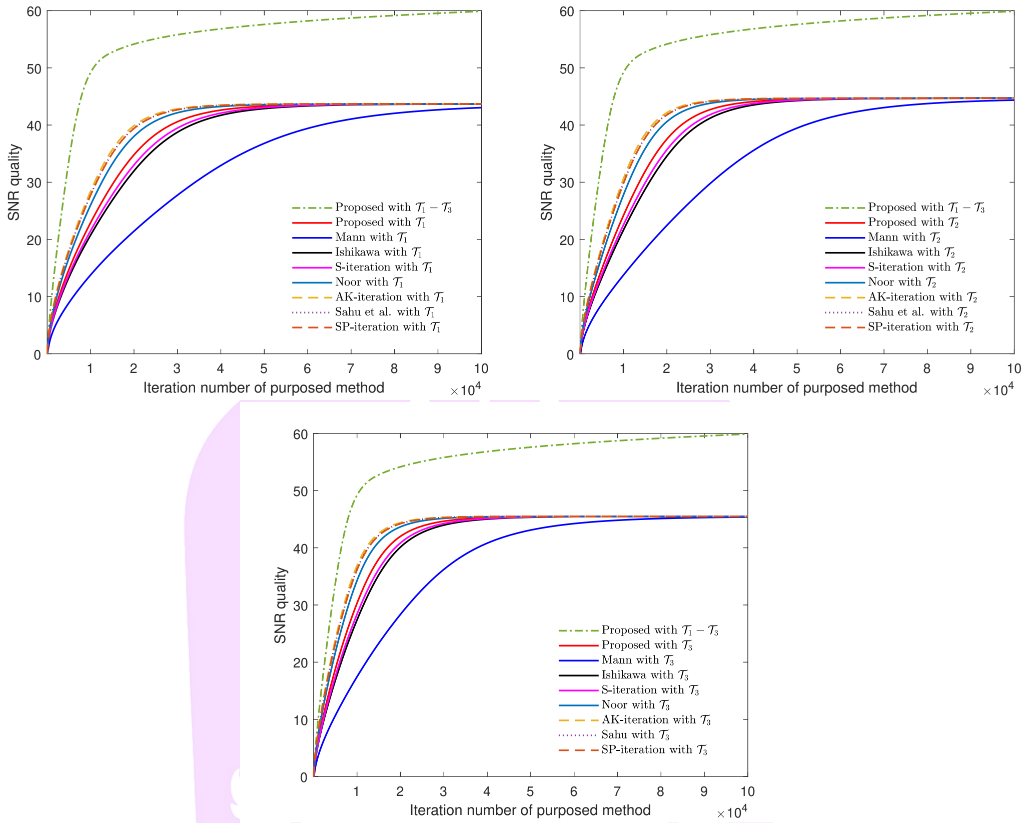

Figure 3.3.18: The SNR plots for all comparing methods comparing the proposed method presented in $100,000^{\text{th}}$ iterations.

It can be seen from Figure 3.3.18 that the SNR quality of Noor, AK, Sahu, et al., and SP iterations are better than the proposed methods. However, the proposed method has been designed to be used with a pair of $\mathcal{T}_j - \mathcal{T}_k$, where $j \neq k$. With this advantage, we found that the proposed method with cases IV - IX is more effective than Noor, AK, Sahu, et al., and SP iterations (see Figure 3.3.16). And when we compare the results of Noor, AK, Sahu, et al., and SP iterations with case V of the proposed method, it is clearly seen that the proposed way is much more effective.

CHAPTER V

CONCLUSIONS

This chapter is all results of this dissertation including lemmas and theorems. We conclude again that what we get from the results.

6.1 Novel Noor iterations technique for solving nonlinear equation

We give a necessary and sufficient condition for the strong convergence of the CT-iteration of continuous functions on an arbitrary interval and improve the rate of convergence compared to previous work. Specifically, our main result shows that CT–iteration converges faster than CP–iteration to the fixed point.

Let C be a closed interval on the real line and $f : C \to C$ given mapping. Then for an arbitrary $x_1 \in C$, the following iteration scheme is studied:

$$\begin{aligned} z_n &= (1-\mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1-\tau_n-\beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1-\gamma_n-\alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n), \ n \geq 1, \end{aligned} \tag{6.1.1}$$

where, $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ are appropriate real sequences in $[0, 1]$. The iterative scheme (6.1.1) is called the CT-iteration for continuous functions.

Theorem 6.1.1 *Let C be a closed interval on the real line (can be unbounded) and let $f : C \to C$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$ such that $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. From an arbitrary initial guess $x_1 \in C$, define the sequence $\{x_n\}$ using (6.1.1). Then $\{x_n\}$ is bounded if and only if it converges to a fixed point of f .*

Corollary 6.1.2 *Let $f : [a, b] \to [a, b]$ be a continuous function. Let $\{\alpha_n\}, \{\beta_n\}$, $\{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1]$. Let $\{x_n\}$ be a sequence generated iteratively by $x_1 \in [a, b]$ and*

$$\begin{aligned} z_n &= (1 - \mu_n)x_n + \mu_n f(x_n), \\ y_n &= (1 - \tau_n - \beta_n)x_n + \tau_n f(x_n) + \beta_n f(z_n), \\ x_{n+1} &= (1 - \gamma_n - \alpha_n)z_n + \gamma_n f(z_n) + \alpha_n f(y_n),\ n \geq 1, \end{aligned}$$

where $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \beta_n < \infty$, $\sum_{n=1}^{\infty} \mu_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} \tau_n < \infty$. Then $\{x_n\}$ converges to a fixed point of f.

Theorem 6.1.3 *Let C be a closed interval on the real line and let $f : C \to C$ be a continuous and nondecreasing function such that $F(f)$ is nonempty and bounded. Let $\{\alpha_n\}, \{\beta_n\}, \{\mu_n\}, \{\gamma_n\}$ and $\{\tau_n\}$ be sequences in $[0, 1)$. For $w_1 = x_1 \in C$, let $\{w_n\}$ and $\{x_n\}$ be sequences defined by the CP–iteration (1.0.5) and the CT–iteration (6.1.1), respectively. If the CP–iteration $\{w_n\}$ converges to $p \in F(f)$, then the CT–iteration $\{x_n\}$ converges to p. Moreover, the CT–iteration (6.1.1) converges faster than the CP–iteration (1.0.5).*

6.2 Numerical reckoning fixed points for nonexpansive mappings via a faster iteration process and its application to constrained minimization problems, split feasibility problems and image deblurring problems

We introduce a new faster iteration process for numerical reckoning fixed points of nonexpansive mappings, where the sequence $\{x_n\}$ is generated iteratively by $x_1 \in C$ and

$$\begin{cases} x_{n+1} = (1-\alpha_n)Tx_n + \alpha_n Ty_n, \\ y_n = (1-\beta_n)Tx_n + \beta_n Tz_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \end{cases} \tag{6.2.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in (0,1).

Theorem 6.2.1 *Let C be a nonempty closed convex subset of a norm space E. Let T be a contraction with a contraction factor $k \in (0,1)$ and fixed point p. Let $\{u_n\}$ be defined by the iteration process (1.0.7) and $\{x_n\}$ by (6.2.1), where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are in $[\epsilon, 1-\epsilon]$ for all $n \in \mathbb{N}$ and for some ϵ in (0, 1). Then $\{x_n\}$ converges faster than $\{u_n\}$. That is, our process (6.2.1) converges faster than (1.0.7).*

Lemma 6.2.2 *Let C be a nonempty closed convex subset of a normed linear space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence defined by (6.2.1) and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(T)$.*

Lemma 6.2.3 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ be a sequence given by (6.2.1) and $F(T) \neq \emptyset$. Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.*

Theorem 6.2.4 *Let E be a real uniformly convex Banach space which satisfies the Opial's condition, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$. Let $\{x_n\}$ be the sequence defined by iteration process (6.2.1). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Theorem 6.2.5 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (6.2.1) and $F(T) \neq \emptyset$. Then $\{x_n\}$ converges to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, where $d(x, F(T)) = \inf\{\|x - p\| : p \in F(T)\}$.*

Theorem 6.2.6 *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let T be a nonexpansive self mapping on C, $\{x_n\}$ defined by (6.2.1) and $F(T) \neq \emptyset$. Let T satisfy Condition (A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 6.2.7 *Let C be a closed convex subset of a Hilbert space H and T a convex and differentiable function on an open set D containing the set C. Assume that $\bigtriangledown T$ is an L–Lipschitz operator on D, $\mu \in (0, 2/L)$ and minimizers of T relative to the set C exist. For a given $x_1 \in C$, let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)P_C(I - \mu \bigtriangledown T)x_n + \alpha_n P_C(I - \mu \bigtriangledown T)y_n, \\ y_n = (1-\beta_n)P_C(I - \mu \bigtriangledown T)x_n + \beta_n P_C(I - \mu \bigtriangledown T)z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n P_C(I - \mu \bigtriangledown T)x_n, \ n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some $\delta \in (0,1)$. Then $\{x_n\}$ converges weakly to a minimizer of T.

Theorem 6.2.8 *Let H be a real Hilbert space. Let $A : H \to H$ be a bounded linear operator and $b \in H$. Let $\{x_n\}$ be a sequence generated by $C_1 = H$, $x_0 \in H$ and*

$$\begin{cases} x_{n+1} = (1-\alpha_n)\left(x_n - \mu A^T(Ax_n - b)\right) + \alpha_n \left(y_n - \mu A^T(Ay_n - b)\right), \\ y_n = (1-\beta_n)\left(x_n - \mu A^T(Ax_n - b)\right) + \beta_n \left(z_n - \mu A^T(Az_n - b)\right), \\ z_n = (1-\gamma_n)x_n + \gamma_n \left(x_n - \mu A^T(Ax_n - b)\right), \ n \in \mathbb{N}, \end{cases}$$

where $\mu \subset \left(0, \frac{2}{||A||_2^2}\right)$ and $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0,1)$. Then $\{x_n\}$ converges weakly to its solution.

Theorem 6.2.9 *Assume that $SFP(C,T)$ is consistent. Suppose $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[\delta, 1-\delta]$ for all $n \in \mathbb{N}$ and for some δ in $(0,1)$. Let $\{x_n\}$ be a sequence in C generated by*

$$\begin{cases} x_{n+1} = (1-\alpha_n)P_C(I-\mu \bigtriangledown q)x_n + \alpha_n P_C(I-\mu \bigtriangledown q)y_n, \\ y_n = (1-\beta_n)P_C(I-\mu \bigtriangledown q)x_n + \beta_n P_C(I-\mu \bigtriangledown q)z_n, \\ z_n = (1-\gamma_n)x_n + \gamma_n P_C(I-\mu \bigtriangledown q)x_n, \; n \in \mathbb{N}, \end{cases}$$

where $0 < \mu < 2/\|T\|^2$. Then $\{x_n\}$ converges weakly to a solution of $SFP(C,T)$.

6.3 New iterative methods for nonlinear operators as concerns convex programming applicable in differential problems, image deblurring and signal recovering problems

Using sunny nonexpansive retractions which are different from the metric projection in Banach spaces, a new type of study regarding the iterative methods in view of two quasi-nonexpansive nonself mappings is presented. We also give the convergence analysis for the proposed method in the background of uniformly convex Banach spaces. Moreover, we apply our results to find solutions of common zeros of accretive operators, convexly constrained least square problems and convex minimization problems.

Algorithm 1 : Three–step iterative methods of two quasi–nonexpansive nonself mappings using sunny nonexpansive retractions

initialization: $\eta_n, \vartheta_n, \zeta_n \in (0,1), u_1 \in \mathcal{K}$ and $n = 1$.
while stopping criterion not met **do**

$$\begin{aligned} w_n &= \mathcal{Q}_{\mathcal{K}}\left((1-\zeta_n)u_n + \zeta_n \mathcal{S}u_n\right), \\ z_n &= \mathcal{Q}_{\mathcal{K}}\left((1-\vartheta_n)\mathcal{T}u_n + \vartheta_n \mathcal{S}w_n\right), \\ u_{n+1} &= \mathcal{Q}_{\mathcal{K}}\left((1-\eta_n)\mathcal{S}z_n + \eta_n \mathcal{T}w_n\right). \end{aligned}$$

end

Theorem 6.3.1 *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{E}$ with $\mathcal{Q}_{\mathcal{K}}$ as the sunny nonexpansive retraction. Let $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{E}$ be quasi-nonexpansive mappings with $\Psi \neq \emptyset$. Let $\{\eta_n\}$,$\{\vartheta_n\}$, $\{\zeta_n\}$ be sequences of real numbers such that $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. From an arbitrary $u_1 \in \mathcal{K}$, $\mathcal{P}_{\Psi}(u_1) = u_*$, define the sequence $\{u_n\}$ by Algorithm 1. Then, we have the following:*

(i) *$\{u_n\}$ is in a closed convex bounded set $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$, where λ is a constant in $(0, \infty)$ such that $\|u_1 - u_*\| \leq \lambda$.*

(ii) *If $\mathcal{S}$ is uniformly continuous, then $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$ and $\lim_{n\to\infty} \|u_n - \mathcal{T}u_n\| = 0$.*

(iii) *If $\mathcal{E}$ fulfills the Opial's condition and $\mathcal{I} - \mathcal{S}$ and $\mathcal{I} - \mathcal{T}$ are demiclosed at 0, then $\{u_n\}$ converges weakly to an element of $\Psi \cap \mathcal{B}_\lambda[u_*]$.*

Theorem 6.3.2 *Let $\mathcal{K}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathcal{E}$ with $\mathcal{Q}_{\mathcal{K}}$ as the sunny nonexpansive retraction. Let $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{E}$ be nonexpansive mappings with $\psi \neq \emptyset$. Let $\{\eta_n\}$,$\{\vartheta_n\}$, $\{\zeta_n\}$ be sequences of real numbers such that $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. From an arbitrary $u_1 \in \mathcal{K}$, $\mathcal{P}_{\Psi}(u_1) = u_*$, define the sequence $\{u_n\}$ by Algorithm 1. Then, we have the following:*

(i) *$\{u_n\}$ is in a closed convex bounded set $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$ where λ is a constant*

in $(0,\infty)$ *such that* $\|u_1 - u_*\| \leq \lambda$.

(ii) $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$ *and* $\lim_{n\to\infty} \|u_n - \mathcal{T}u_n\| = 0$.

(iii) *If* $\mathcal{E}$ *fulfills the Opial's condition, then* $\{u_n\}$ *converges weakly to an element of* $\Psi \cap \mathcal{B}_\lambda[u_*]$.

If $\mathcal{S}$ and $\mathcal{T}$ are nonexpansive self mappings on a nonempty closed convex subset $\mathcal{K}$ of a real Hilbert space $\mathcal{H}$. As a result of Theorem 6.3.1, we can get the following result.

Corollary 6.3.3 *Let* $\mathcal{K}$ *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$*, let* $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ *be nonexpansive mappings with* $\Psi \neq \emptyset$ *and let* $\{\eta_n\}$,$\{\vartheta_n\}$ *and* $\{\zeta_n\}$ *be sequences of real numbers, for which* $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ *for all* $n \in \mathbb{N}$. *From an arbitrary* $u_1 \in \mathcal{K}$, *define the sequence* $\{u_n\}$ *by :*

$$\begin{cases} w_n = (1-\zeta_n)u_n + \zeta_n \mathcal{S}u_n \\ z_n = (1-\vartheta_n)\mathcal{T}u_n + \vartheta_n \mathcal{S}w_n \\ u_{n+1} = (1-\eta_n)\mathcal{S}z_n + \eta_n \mathcal{T}w_n, \quad \forall n \in \mathbb{N}. \end{cases} \tag{6.3.1}$$

Then, $\{u_n\}$ *converges weakly to an element of* Ψ.

Setting $\mathcal{S} = J_\mu^{\mathcal{A}}$ and $\mathcal{T} = J_\mu^{\mathcal{B}}$, using (6.3.1), we derive its convergence analysis for finding solutions (1.0.13).

Theorem 6.3.4 *Let* $\mathcal{K}$ *be a nonempty closed convex subset of a real uniformly convex Banach space* $\mathcal{E}$ *satisfying the opial's condition. Let* $\mathcal{A} : \mathcal{D}(\mathcal{A}) \subseteq \mathcal{K} \to 2^{\mathcal{E}}$, $\mathcal{B} : \mathcal{D}(\mathcal{B}) \subseteq \mathcal{K} \to 2^{\mathcal{E}}$ *be accretive operators such that* $\overline{\mathcal{D}(\mathcal{A})} \subseteq \mathcal{K} \subseteq \cap_{\mu>0}\mathcal{R}(\mathcal{I}+\mu\mathcal{A})$, $\overline{\mathcal{D}(\mathcal{B})} \subseteq \mathcal{K} \subseteq \cap_{\mu>0}\mathcal{R}(\mathcal{I}+\mu\mathcal{B})$ *and* $\mathcal{A}^{-1}(0) \cap \mathcal{B}^{-1}(0) \neq \emptyset$. *Let* $\{\eta_n\}$, $\{\vartheta_n\}$, $\{\zeta_n\}$ *be sequences of real numbers such that* $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ *for all* $n \in \mathbb{N}$. *Let* $\mu > 0$, $u_1 \in \mathcal{K}$ *and* $\mathcal{P}_{\mathcal{A}^{-1}(0)\cap\mathcal{B}^{-1}(0)}(u_1) =$

u_*. Let $\{u_n\}$ be defined by

$$\begin{cases} w_n = (1-\zeta_n)u_n + \zeta_n J_\mu^{\mathcal{A}} u_n, \\ z_n = (1-\vartheta_n)J_\mu^{\mathcal{B}} u_n + \vartheta_n J_\mu^{\mathcal{A}} w_n, \\ u_{n+1} = (1-\eta_n)J_\mu^{\mathcal{A}} z_n + \eta_n J_\mu^{\mathcal{B}} w_n, \ \forall n \in \mathbb{N}. \end{cases} \tag{6.3.2}$$

Then, we have the following:

(i) $\{u_n\}$ is in a closed convex bounded set $\mathcal{B}_\lambda[u_*] \cap \mathcal{K}$, where λ is a constant in $(0,\infty)$ such that $\|u_1 - u_*\| \leq \lambda$.

(ii) $\lim_{n\to\infty} \|u_n - J_\mu^{\mathcal{A}} u_n\| = 0$ and $\lim_{n\to\infty} \|u_n - J_\mu^{\mathcal{B}} u_n\| = 0$.

(iii) $\{u_n\}$ converges weakly to an element of $\mathcal{A}^{-1}(0) \cap \mathcal{B}^{-1}(0) \cap \mathcal{B}_\lambda[u_*]$.

Proposition 6.3.5 *[1] Let $\mathcal{H}$ be a real Hilbert space, $\mathcal{A} \in \mathcal{B}(\mathcal{H})$ with the adjoint $\mathcal{A}^*$ and $y \in \mathcal{H}$. Let $\mathcal{K}$ be a nonempty closed convex subset of $\mathcal{H}$. Let $b \in \mathcal{H}$ and $\delta \in (0,\infty)$. Then, the following statements are equivalent:*

(i) b solves the following problem:

$$\min_{u\in\mathcal{K}} \frac{1}{2}\|\mathcal{A}u - y\|^2.$$

(ii) $b = \mathcal{P}_{\mathcal{K}}(b - \delta\mathcal{A}^*(\mathcal{A}b - y))$.

(iii) $\langle \mathcal{A}v - \mathcal{A}b, y - \mathcal{A}b\rangle \leq 0$, for all $v \in \mathcal{K}$.

Theorem 6.3.6 *Let $\mathcal{K}$ be a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$, $y, z \in \mathcal{H}$ and $\mathcal{A}, \mathcal{B} \in \mathcal{B}(\mathcal{H})$ such that the solution set of the problem in (3.3.22) is nonempty. Let $\{\eta_n\}$, $\{\vartheta_n\}$ and $\{\zeta_n\}$ be sequences of real numbers such that $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ for all $n \in \mathbb{N}$. Let $u_1 \in \mathcal{H}$, $\mathcal{P}_{argmin_{u\in\mathcal{K}}\,\varphi(u)\,\cap\,argmin_{u\in\mathcal{K}}\,\psi(u)}(u_1) = u_*$ and $\delta \in (0, 2min\{\frac{1}{\|\mathcal{A}\|^2}, \frac{1}{\|\mathcal{B}\|^2}\})$.*

From an arbitrary $u_1 \in \mathcal{K}$, *define the sequence* $\{u_n\}$ *by*

$$\begin{cases} w_n = (1-\zeta_n)u_n + \zeta_n \mathcal{S}u_n, \\ z_n = (1-\vartheta_n)\mathcal{T}u_n + \vartheta_n \mathcal{S}w_n, \\ u_{n+1} = (1-\eta_n)\mathcal{S}z_n + \eta_n \mathcal{T}w_n, \forall n \in \mathbb{N}, \end{cases} \tag{6.3.3}$$

where $\mathcal{S}, \mathcal{T} : \mathcal{K} \to \mathcal{K}$ *defined by* $\mathcal{S}u = \mathcal{P}_{\mathcal{K}}(u - \delta\mathcal{A}^*(\mathcal{A}u - y))$ *and* $\mathcal{T}u = \mathcal{P}_{\mathcal{K}}(u - \delta\mathcal{B}^*(\mathcal{B}u - z))$ *for all* $u \in \mathcal{K}$. *Then, we have the following:*

(i) $\{u_n\}$ *is in the closed ball* $\mathcal{B}_\lambda[u_*]$, *where* λ *is a constant in* $(0,\infty)$ *such that* $\|u_1 - u_*\| \leq \lambda$.

(ii) $\lim_{n\to\infty} \|u_n - \mathcal{S}u_n\| = 0$ *and* $\lim_{n\to\infty} \|u_n - \mathcal{T}u_n\| = 0$.

(iii) $\{u_n\}$ *converges weakly to an element of* $argmin_{u\in\mathcal{K}}\, \varphi(u) \cap argmin_{u\in\mathcal{K}}\, \psi(u) \cap d\, \mathcal{B}_\lambda[u_*]$.

Theorem 6.3.7 *Let* $\mathcal{K}$ *be a nonempty closed convex subset of a real Hilbert space* $\mathcal{H}$. *Let* $g_1, g_2 \in \Gamma_0\mathcal{H}$ *such that the solution set of the problem in (3.3.24) is nonempty. Let* $\{\eta_n\}$, $\{\vartheta_n\}$ *and* $\{\zeta_n\}$ *be sequences of real numbers such that* $0 < c_1 \leq \eta_n \leq \hat{c}_1 < 1$, $0 < c_2 \leq \vartheta_n \leq \hat{c}_2 < 1$, $0 < c_3 \leq \zeta_n \leq \hat{c}_3 < 1$ *for all* $n \in \mathbb{N}$. *Let* $\mu > 0$, $u_1 \in \mathcal{H}$ *and* $\mathcal{P}_{\partial g_1^{-1}(0) \cap g_2^{-1}(0)}(u_1) = u_*$. *From an arbitrary* $u_1 \in \mathcal{K}$, *define the sequence* $\{u_n\}$ *by*

$$\begin{cases} w_n = (1-\zeta_n)u_n + \zeta_n prox_{\mu g_1}(u_n), \\ z_n = (1-\vartheta_n)prox_{\mu g_2}(u_n) + \vartheta_n prox_{\mu g_1}(w_n), \\ u_{n+1} = (1-\eta_n)prox_{\mu g_1}(z_n) + \eta_n prox_{\mu g_2}(w_n), \ \forall n \in \mathbb{N}. \end{cases} \tag{6.3.4}$$

Then, we have the following:

(i) $\{u_n\}$ *is in the closed ball* $\mathcal{B}_\lambda[u_*]$ *where* λ *is a constant in* $(0,\infty)$ *such that* $\|u_1 - u_*\| \leq \lambda$.

(ii) $\lim_{n\to\infty} \|u_n - prox_{\mu g_1}(u_n)\| = 0$ *and* $\lim_{n\to\infty} \|u_n - prox_{\mu g_2}(u_n)\| = 0$.

(iii) $\{u_n\}$ *converges weakly to an element of* $\partial g_1^{-1}(0) \cap g_2^{-1}(0) \cap \mathcal{B}_\lambda[u_*]$.

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Education Background

2010	M.Sc. (Mathematics), Naresuan University, Phitsanulok, Thailand
2005	Grad. Dip. (Teaching Profession), Naresuan University, Phitsanulok, Thailand
2004	B.Sc. (Mathematics), Naresuan University, Phitsanulok, Thailand

Publications

1. **Chairatsiripong, C.**, Yambangwai, D., and Thianwan, T. (2020). Numerical reckoning fixed points for nonexpansive mappings via a faster iteration process and its application to constrained minimization problems, split feasibility problems and image deblurring problems. Thai Journal of Mathematics, 18(3), 1323 - 1342.

2. **Chairatsiripong, C.**, and Thianwan, T. (2021). Novel Noor iterations technique for solving nonlinear equations. AIMS Mathematics, 7(6), 10958 - 10976.

3. **Chairatsiripong, C.**, Yambangwai, D., and Thianwan, T. (2022). New iterative methods for nonlinear operators as concerns

convex programming applicable in differential problems, image deblurring and signal recovering problems. Mathematical Methods in the Applied Sciences, 46(3), 3332–3355.

Scholarship

1. **National Research Council of Thailand (NRCT)** (2021). RGJ Ph.D. Grant number N41A640216

Short–term research

1. **School of Computing and Engineering, University of Derby, Derby, UK** (1 March – 31 May 2022). Collaborate with Associate Professor Dr.Ovidiu Bagdasar
2. **Faculty of Mathematics, University of Wien, Vienna, Austria** (1 July – 31 September 2023). Collaborate with Professor Dr.Radu Ioan Bot

Conference presentations

1. **Chairatsiripong, C.** (2021). Some new iterative techniques for the problems involving image restoration,Virtual International Conference on Soft Computing, Optimization Theory and Applications, SCOTA 2021, Mersa, Ranchi-835215, Jharkhand, India.
2. **Chairatsiripong, C.** (2021). A new faster proximal method for solving the problem of image deblurring, The international workshop on digital image processing and machine learning (IWDM 2021), Chiang Mai, Thailand.

3. **Chairatsiripong, C.** (2021). A new iteration technique for nonlinear operators as concerns convex programming and applicable in signal processing, The 24th international annual symposium on computational science and engineering, Kasetsart University, Bangkok, Thailand.

4. **Chairatsiripong, C.** (2022). Novel Noor iterations techniqe for solving nonlinear equations, The 2022 National RGJ and RRI Conference, Thailand Research Expo 2022, Bangkok, Thailand.

5. **Chairatsiripong, C.** (2022). Common fixed point of the inertial new 3–step algorithm applied to polynomiography for permutation matrices, International conference on Digital Image Processing and Machine Learning (ICDIPML 2022), Phayao, Thailand.

6. **Chairatsiripong, C.** (2023). Modified Jungck S–iteration to generate polynomiographs applicable in nonlinear complex functions, The 14$^{\text{th}}$ National Science Research Conference (SRC 14), Phayao, Thailand.