CONVERGENCE THEOREMS FOR THE SPLIT FEASIBILITY PROBLEM

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Convergence Theorems For The Split Feasibility Problem

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เรื่อง: ทฤษฎีบทการลู่เข้าสำหรับปัญหาความเป็นไปได้แยกส่วน

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คำสำคัญ: ปัญหาความเป็นไปได้แยกส่วน, ขั้นตอนวิธีการฉาย, ปริภูมิฮิลเบิร์ต, วิธีการเกรเดียนต์

บทคัดย่อ

ปัญหาที่สำคัญและน่าสนใจในทฤษฎีค่าเหมาะสม คือ ปัญหาความเป็นไปได้แยกส่วน ปัญหาดังกล่าวนี้ได้รับความสนใจอย่างมาก เนื่องจากปัญหาจำนวนมากในทางวิทยาศาสตร์และวิทยาศาสตร์ประยุกต์สามารถกำหนดรูปแบบเป็นปัญหาความเป็นไปได้แยกส่วน เช่น การประมวลผลสัญญาณและการกู้คืนภาพ ในงานวิจัยนี้ได้มีการปรับปรุงขั้นตอนวิธีการฉายและขั้นตอนวิธีการฉายแบบผ่อนปรน สำหรับแก้ปัญหาความเป็นไปได้แยกส่วนในขอบเขตของปริภูมิฮิลเบิร์ต ข้อได้เปรียบที่สำคัญของระเบียบวิธีการนี้ คือการสร้างลำดับของวิธีทำซ้ำไม่จำเป็นต้องใช้นอร์มของตัวดำเนินการในการคำนวณ และการฉายเมตริกไปยังเซตย่อยของปริภูมิฮิลเบิร์ตซึ่งมีสูตรที่แน่นอนได้ถูกรวมเข้าไปในวิธีการทำซ้ำ จากนั้นได้มีการพิสูจน์การลู่เข้าแบบอ่อนและแบบเข้มภายใต้เงื่อนไขที่เหมาะสม สุดท้ายนี้ได้มีการศึกษาการทดลองเชิงตัวเลขเพื่อแสดงให้เห็นถึงประสิทธิภาพของขั้นตอนวิธีที่ถูกนำเสนอ ผลลัพธ์ที่ได้สามารถปรับปรุงและขยายผลลัพธ์ของงานวิจัยที่เกี่ยวข้อง

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ABSTRACT

One of the most important and interesting problems in optimization theory is the split feasibility problem. This problem has been intensively investigated since many problems in sciences and applied sciences can be reformulated as the split feasibility problem such as signal processing and image reconstruction.

In this research, the modified projection algorithm and the relaxed projection algorithm for solving the split feasibility problems are studied in the framework of Hilbert spaces. The main advantage of the proposed method is that the operator norms do not require in computing the sequences and that the metric projections onto subsets of Hilbert spaces which have exact formulas are involved in iterative methods. Then both weak and strong convergence theorems are proved under some suitable conditions. Finally, numerical experiments are investigated to show the efficiency of the proposed algorithms. The obtain results improve and extend the corresponding results in the literature.

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CHAPTER I

INTRODUCTION

In optimization theory, a major problem is the split feasibility problem (SFP). It can be a unified model for many practical problems such as in signal processing and image reconstruction, intensity - modulated radiation therapy and many other applied fields. To be more precise, the split feasibility problem includes, as special cases, the convex minimization problem, the linear inverse problem, the fixed point problem of some nonlinear operators. The regularization technique is a powerful tool in handling for solving such problem in some certain spaces. Censor - Elfving [8] introduced a notion of the split feasibility problem (SFP), which is to find an element of a closed convex subset of the Euclidean space. In 2002, Byrne [6] proposed iterative oblique projection onto convex sets and the split feasibity problem. L*ó*pez et al. [24] proposed the iterative scheme for the split feasibility problem without prior knowledge of operator norms such that projections onto half - spaces. However, their algorithm has only weak convergence in the setting of infinite - dimensional Hilbert spaces. In 2010 Xu HK. [46] proposed iterative methods for the split feasibility problem in infinite - dimensional Hilbert spaces. Zhang et al. [50] proposed a self - adaptive projection method for solving the multiple - sets split feasibility problem in Hilbert spaces. Recently, He et al. [19] introduced a new relaxed CQ algorithm for solving the MSFP, and proved the strong convergence by using the Halpern - type algorithm in real Hilbert spaces. In 2005, Qu and Xiu [33] modified the relaxed CQ algorithm by adopting Armijo - line searches in Euclidean spaces. Next, Gibali et al. [16] extended the results of Qu and Xiu [33] to Hilbert spaces. Korpelevich [22] and Antipin [1] proposed the extragradient method which is a classical two - step method. Recently, Dong et al. [14] proposed the modified projection and contraction methods and their relaxation variants to solve the split feasibility problem (SFP).

It is therefore the main objective in this research to design new algorithms for solving the split feasibility problem in Hilbert spaces. We prove the convergence theorems under some suitable conditions. We also provide some numerical examples to support our main result. The main results established in this research can improve and generalize the corresponding results in this area.

CHAPTER II

REVIEW OF RELATED LITERATURE AND RESEARCH

We study the Split Feasibility Problem (SFP) which is described as the following form:

$$\text{find a point } x^* \in C \text{ such that } Ax^* \in Q \tag{2.1.1}$$

where C and Q are nonempty closed convex subsets of real Hilbert spaces H_1 and H_2, respectively, and $A : H_1 \to H_2$ is a bounded linear operator. This problem was first introduced, in Euclidean spaces, by Censor and Elfving [8]. The SFP relates to an inverse problem in intensity - modulated radiation therapy (IMRT) in the field of medical care and the LASSO problem in signal recovery and image processing. Throughout this work, we assume that SFP (2.1.1) is consistent and denote the solution set by S.

Byrne [5, 6] introduced CQ algorithm which generates a sequence $\{x_n\}$ as follows:

$$x_{n+1} = P_C(x_n - \tau_n A^*(I - P_Q)Ax_n) \tag{2.1.2}$$

where the stepsize $\tau_n \in (0, 2/\|A\|^2)$, A^* is the adjoint operator of A, P_C and P_Q are the metric projections onto C and Q, respectively. It is seen that if the metric projections onto C and Q are easily calculated, then the total cost of computation is not great. However, in some cases it is impossible or needs too much work to exactly compute the metric projection. The determination of the stepsize depends on the operator norm which computation (or at least estimate) is not an easy task.

In practical applications, the sets C and Q are usually the level sets of convex functions which are given by

$$C = \{x \in H_1 : c(x) \le 0\} \text{ and } Q = \{y \in H_2 : q(y) \le 0\} \tag{2.1.3}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex functions and subdifferential functions on H_1 and H_2, respectively, and that ∂c and ∂q are bounded operators (*i.e.* bounded on bounded sets).

Fukushima [15] proposed a relaxed projection algorithm for solving variational inequality and the theoretical analysis and numerical experiment showed the efficiency of the proposed method.

In 2004, Yang [49] introduced the relaxed CQ algorithm, by replacing P_C and P_Q by P_{C_n} and P_{Q_n}, respectively. Here C_n and Q_n are given by

$$C_n = \{x \in H_1 : c(x_n) \leq \langle \xi_n, x_n - x \rangle\}, \tag{2.1.4}$$

where $\xi_n \in \partial c(x_n)$ and

$$Q_n = \{y \in H_2 : q(Ax_n) \leq \langle \zeta_n, Ax_n - y \rangle\}, \tag{2.1.5}$$

where $\zeta_n \in \partial q(Ax_n)$. It is easy to see that $C_n \supset C$ and $Q_n \supset Q$ for every $n \geq 1$. Moreover the projections onto half - spaces C_n and Q_n have closed forms. In what follows, define

$$f_n(x) = \frac{1}{2}\|(I - P_{Q_n})Ax\|^2, \; n \geq 1 \tag{2.1.6}$$

where Q_n is given as in (2.1.5). In this case, we then have

$$\nabla f_n(x) = A^*(I - P_{Q_n})Ax. \tag{2.1.7}$$

Since these projections are easily calculated, this method appears to be very practical. In fact, Yang [49] introduced a relaxed CQ algorithm in a finite - dimensional Hilbert space as follows:

$$x_{n+1} = P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{2.1.8}$$

where $\tau_n \in (0, 2/\|A\|^2)$. We note that to compute the norm of A turns out to be complicated and costly. Especially, A is a dense matrix and has a large dimension.

In 2005, Yang [48] suggested a new way to select the stepsize τ_n which is defined as follows:

$$\tau_n = \frac{\rho_n}{\|\nabla f_n(x_n)\|} \tag{2.1.9}$$

where $\{\rho_n\}$ is a sequence of positive real numbers satisfying

$$\sum_{n=1}^{\infty} \rho_n = \infty, \qquad \sum_{n=1}^{\infty} \rho_n^2 < +\infty. \tag{2.1.10}$$

However, in this case, the stepsize (2.1.9) requires A to have a full column rank.

In 2012, López et al. [24], to overcome this difficulty, introduced a new way to select the step - size and also practised this way of selecting stepsizes for variants of the CQ algorithm, include a relaxed CQ algorithm. They introduced the stepsize τ_n which is defined as follows:

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|\nabla f_n(x_n)\|^2}, \tag{2.1.11}$$

where $\{\rho_n\}$ is a sequence in $(0, 4)$ such that $\inf_{n\in\mathbb{N}} \rho_n(4 - \rho_n) > 0$. It was proved that the sequence $\{x_n\}$ generated by (2.1.8) with the stepsize defined by (2.1.11) converges weakly to a solution of SFP.

They also studied the Halpern - type algorithm to guarantee the strong convergence as follows:

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{2.1.12}$$

where $u \in H_1$ is fixed and τ_n is defined by (2.1.11). If the sequence $\{x_n\}$ is generated by (2.1.12), then it is converges strongly to $P_S u$ provide $\alpha_n \to 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

In 2005, Qu and Xiu [33] modified the relaxed CQ algorithm by adopting Armijo-line searches in Euclidean spaces. Subsequently, Gibali et al. [16] extended the results of Qu and Xiu [33] to Hilbert spaces as follows:

$$x_{n+1} = P_{C_n}(x_n - \tau_n \nabla f_n(y_n)) \tag{2.1.13}$$

$$y_n = P_{C_n}(x_n - \tau_n \nabla f_n(x_n)), \tag{2.1.14}$$

where $\tau_n = \gamma \ell^{m_n}$ and m_n is the smallest nonnegative integer m, and $\gamma > 0$, $\ell \in (0,1)$ and $\mu \in (0,1)$ such that

$$\tau_n \|\nabla f_n(x_n) - \nabla f_n(y_n)\| \leq \mu \|x_n - y_n\|. \tag{2.1.15}$$

They proved that $\{x_n\}$ weakly converges to a solution of SFP. Various iterative methods have been constructed to solve the SFP; see [12, 43, 46, 51].

Korpelevich [22] and Antipin [1] proposed the following extragradient method:

$$\begin{aligned} y_n &= P_C(x_n - \tau_n F(x_n)) \\ x_{n+1} &= P_C(x_n - \tau_n F(y_n)) \end{aligned} \tag{2.1.16}$$

where $F = A^*(I - P_Q)A$ and the fixed stepsize $\tau_n \in (0, \frac{1}{\|F\|})$, which is a classical two - step method. The second one is to select self - adaptively the stepsize $\tau_n > 0$ such that

$$\tau_n \|F(x_n) - F(y_n)\| \leq \mu \|x_n - y_n\|, \ \forall \mu \in (0,1). \tag{2.1.17}$$

In [42], Tseng proposed the following extragradient methods:

$$\begin{aligned} y_n &= P_C(x_n - \tau_n F(x_n)) \\ x_{n+1} &= y_n + \tau_n (F(x_n) - F(y_n)) \end{aligned} \tag{2.1.18}$$

where $\tau_n \in (0, \frac{1}{\|F\|})$ or $\{\tau_n\}$ is selected self-adaptively. Subsequently, Zhao et al. [51] used Tseng's method (2.1.18) to solve the SFP. Recently, Dong et al. [14] proposed the modified projection and contraction methods and their relaxation variants to solve the SFP as follows:

Algorithm 2.1.1. *For any $\sigma > 0, \rho \in (0,1)$ and $\mu \in (0,1)$, take arbitrarily $x_1 \in \mathbb{R}^N$ and let*

$$y_n = P_C(x_n - \tau_n F(x_n)) \tag{2.1.19}$$

where $\tau_n = \sigma\rho^{m_n}$ *and* m_n *is the smallest nonnegative integer such that*

$$\tau_n\|F(x_n) - F(y_n)\| \leq \mu\|x_n - y_n\|. \tag{2.1.20}$$

Define

$$x_{n+1} = x_n - \gamma\phi_n d(x_n, y_n) \tag{2.1.21}$$

where $\gamma \in (0, 2)$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n(F(x_n) - F(y_n)) \tag{2.1.22}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n\|(I - P_Q)A(y_n)\|^2}{\|d(x_n, y_n)\|^2}. \tag{2.1.23}$$

It was proved that the sequence generated by Algorithm 2.1.1 converges to a solution in SFP.

In our research, we combine the gradient method and the relaxed CQ algorithm called the gradient - CQ algorithm with a new stepsize for solving SFP in Hilbert spaces. Moreover, motivated by Dong et al. [14], we propose the modified projection and contraction methods including its relaxation to solve the SFP in real Hilbert spaces. We then prove weak and strong convergence theorems under some suitable conditions. Finally, we present numerical examples and give comparisons to the relaxed CQ algorithms of Yang [49], López et al. [24], Gibali et al. [16] and Dong et al. [14]. It is shown that our proposed algorithms have a number of advantage (in terms of number of iterations and CPU time) over these methods in computing through numerical experiments.

CHAPTER III

PRELIMINARIES

3.1 Fundamentals

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel.

Definition 3.1.1. (Metric space) Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a function. Then d is called a *metric* on X if the following properties hold:

1. $d(x, y) \geq 0$ for all $x, y \in X$;
2. $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;
3. $d(x, y) = d(y, x)$ for all $x, y \in X$;
4. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

The value of metric d at (x, y), we write $d(x, y)$, is called *distance* between x and y, and the ordered pair (X, d) is called a *metric space*.

Example 3.1.2. *In real line $\mathbb{R}$, define*

$$d(x, y) = |x - y| \tag{3.1.1}$$

for all $x, y \in \mathbb{R}$.Then $(\mathbb{R}, d)$ is a metric space.

Example 3.1.3. *In euclidean plane $\mathbb{R}^2$, define*

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2} \tag{3.1.2}$$

where $x = (\xi_1, \xi_2), y = (\eta_1, \eta_2) \in \mathbb{R}^2$. Then $(\mathbb{R}^2, d)$ is a metric space.

Example 3.1.4. *In euclidean space $\mathbb{R}^n$, define*

$$d(x, y) = \sqrt{(\xi_1 - \eta_1)^2 + (\xi_2 - \eta_2)^2 + (\xi_3 - \eta_3)^2 + ... + (\xi_n - \eta_n)^2} \tag{3.1.3}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$. Then $(\mathbb{R}^n, d)$ is a metric space.

Example 3.1.5. *Let X be the set of all bounded sequences of complex numbers; that is every element of X is a complex sequence*

$$x = (\xi_1, \xi_2, ...)$$

such that $|\xi_j| \leq c_x$ for all $j = 1, 2, ...$ and c_x is a real number which may depend on x, but does not depend on j and define

$$d(x, y) = \sup_{j \in \mathbb{N}} |\xi_j - \eta_j| \tag{3.1.4}$$

where $y = (\eta_j) \in X$ and $\mathbb{N} = 1, 2, ...$ Then (X, d) is a metric space.

Definition 3.1.6. (Open and Closed sets) Let (X, d) be a metric space. A subset $U \subseteq X$ is open if for every $x \in X$ there exists $r > 0$ such that $B(x, r) \subseteq U$. A set U is closed if its complement, $X \setminus U$, is open.

Definition 3.1.7. (Convergent sequence) A sequence $\{x_n\}$ in a metric space X is said to be convergent to $x \in \mathbb{R}$ if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > N$. In this case, we write $x_n \to x$.

Definition 3.1.8. (Cauchy sequence) A sequence $\{x_n\}$ in a metric space X is said to be *Cauchy* if for each $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for all $m, n > N$.

Theorem 3.1.9. *Let M be a nonempty subset of a metric space X. Then M is closed if and only if there exists a sequence $\{x_n\} \subseteq M$ and $x_n \to x$ implies that $x \in M$.*

Definition 3.1.10. (Bounded sequence) A sequence $\{x_n\}$ in X is bounded if there exists $M > 0$ such that $\|x_n\| \leq M$ for all $n \in \mathbb{N}$.

Definition 3.1.11. (Nonexpansive mapping) Let (X, d) be a metric space. Then a map $T : X \to X$ is said to be *nonexpansive* if

$$d(T(x), T(y)) \leq d(x, y)$$

for all $x, y \in X$.

Definition 3.1.12. (Contractive mapping) Let (X, d) be a metric space. Then a map $T : X \to X$ is said to be *contractive* if there exists $k \in [0, 1)$ such that

$$d(T(x), T(y)) \leq kd(x, y)$$

for all $x, y \in X$.

Definition 3.1.13. (Fixed point) Let X be a nonempty set and $T : X \to X$. We say that $x \in X$ is a fixed point of T if

$$T(x) = x \tag{3.1.5}$$

and denote by $Fix(T)$ the set of all fixed points of T.

Theorem 3.1.14. (The Banach contraction principle) *Let X be a complete metric space and let T be a contraction of X into itself. Then T has a unique fixed point.*

Definition 3.1.15. (Vector space) A vector space or linear space X over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) is a set X together with an internal binary operation $(+)$ called addition and a scalar multiplication carrying (α, x) in $\mathbb{K} \times X$ to αx in X satisfying the following statements for all $x, y, z \in X$ and $\alpha, \beta \in \mathbb{K}$:

1. $x + y = y + x$;
2. $(x + y) + z = x + (y + z)$;
3. there exists an element $0 \in X$ call the *zero vector* of X such that $x + 0 = x$ for all $x \in X$;
4. for every element $x \in X$, there exists an element $-x \in X$ called *the additive inverse* or *the negative* of x such that $x + (-x) = 0$;
5. $\alpha(x + y) = \alpha x + \alpha y$;
6. $(\alpha + \beta)x = \alpha x + \beta y$;
7. $(\alpha\beta)x = \alpha(\beta x)$;
8. $1 \cdot x = x$.

The elements of a vector space X are called vectors, and the elements of $\mathbb{K}$ are called scalars.

Example 3.1.16. *In euclidean space* $\mathbb{R}^n$*, define*

$$\begin{aligned} x+y &= (\xi_1+\eta_1, \xi_2+\eta_2, \xi_3+\eta_3, ..., \xi_n+\eta_n) \\ \alpha x &= (\alpha\xi_1, \alpha\xi_2, \alpha\xi_3, ..., \alpha\xi_n) \end{aligned}$$

where $x = (\xi_1, \xi_2, \xi_3, ..., \xi_n), y = (\eta_1, \eta_2, \eta_3, ..., \eta_n) \in \mathbb{R}^n$ *and* $\alpha \in \mathbb{R}$*. Then, space* $\mathbb{R}^n$ *is a real vector space.*

Definition 3.1.17. (Convex set) Let C be a subset of a linear space X. Then C is said to be *convex* if $(1-\lambda)x + \lambda y \in C$ for all $x, y \in C$ and all scalar $\lambda \in [0, 1]$.

Example 3.1.18. *1. Every subspace of vector space is convex.*

2. $\overline{B}(x; r) = \{x : \|x\| \leq r\}$ *is convex.*

3. $[0, 1]^N = [1, 0] \times [1, 0] \times ... \times [1, 0]$ *is convex in* $\mathbb{R}^n$*.*

Proposition 3.1.19. *Let* C *be a subset of a linear space* X*. Then* C *is convex if and only if* $\lambda_1 x_1 + \lambda_2 x_2 + ... + \lambda_n x_n \in C$ *for any finite set* $\{x_1, x_2, ..., x_n\} \subseteq C$ *and scalars* $\lambda_i \geq 0$ *with* $\lambda_1 + \lambda_2 + ... + \lambda_n = 1$*.*

Definition 3.1.20. (Convex function) Let X be a linear space and $f : X \to (-\infty, \infty]$ be a function. Then f is said to be *convex* if $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ for all $x, y \in X$ and $\lambda \in [0, 1]$.

Definition 3.1.21. (Proper function) Let function $f : X \to (-\infty, \infty]$. Then f is said to be *proper* if there exists $x \in X$ with $f(x) < \infty$.

Example 3.1.22. *1.* $f(x) = |x|^p$ *where* $p \geq 1$ *is a convex function in* $\mathbb{R}$*.*

2. $f(x) = x^3 - x^2$ *is a convex function in* $[\frac{1}{3}, \infty)$*.*

3. $f(x) = x \log x$ *is a convex function in* $\mathbb{R}^+$*.*

Definition 3.1.23. (Normed space) Let X be a norm linear space over field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and $\|\cdot\| : X \to \mathbb{R}^+$ be a function. Then $\|\cdot\|$ is said to be a norm if the following properties hold:

1. $\|x\| \geq 0$, and $\|x\| = 0 \Leftrightarrow x = 0$;
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$;

3. $\|x+y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

The ordered pair $(X, \|\cdot\|)$ is called a *normed space*.

Example 3.1.24. $\mathbb{R}^n$ *is a normed space with the following norms:*

$$\begin{aligned} \|x\|_1 &= \sum_{i=1}^{n} |x_i| \textit{ for all } x = (x_1, x_2, .., x_n) \in \mathbb{R}^n; \\ \|x\|_p &= \Big(\sum_{i=1}^{n} |x_i|^p\Big)^{1/p} \textit{ for all } x = (x_1, x_2, .., x_n) \in \mathbb{R}^n \textit{ and } p \in (1, \infty); \\ \|x\|_\infty &= \max_{1 \leq i \leq n} |x_i| \textit{ for all } x = (x_1, x_2, .., x_n) \in \mathbb{R}^n. \end{aligned}$$

Example 3.1.25. *Let $X = l_1$, the linear space whose elements consist of all absolutely convergent sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),*

$$l_1 = \{x : x = (x_1, x_2, ..., x_i, ...) \textit{ and } \sum_{i=1}^{\infty} |x_i| < \infty\}.$$

Then l_1 is a normed space with the norm defined by $\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$.

Example 3.1.26. *Let $X = l_p$ $(1 < p < \infty)$ be the linear space whose elements consist of all p - summable sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),*

$$l_p = \{x : x = (x_1, x_2, ..., x_i, ...) \textit{ and } \sum_{i=1}^{\infty} |x_i|^p < \infty\}.$$

Then l_p is a normed space with the norm defined by $\|x\|_p = (\sum_{i=1}^{\infty} |x_i|^p)^{1/p}$.

Example 3.1.27. *Let $X = l_\infty$, the linear space whose elements consist of all bounded sequences $(x_1, x_2, ..., x_i, ...)$ of scalars ($\mathbb{R}$ or $\mathbb{C}$),*

$$l_\infty = \{x : x = (x_1, x_2, ..., x_i, ...) \textit{ and } \{x_i\}_{i=1}^{\infty} \textit{ is bounded}\}.$$

Then l_∞ is a normed space with the norm defined by $\|x\|_\infty = \sup_{i \in \mathbb{N}} |x_i|$.

Definition 3.1.28. (Completeness) The space X is said to be *complete* if every Cauchy sequence in X converges.

Example 3.1.29. *The euclidean space* $\mathbb{R}^n$ *is complete with*

$$d(x,y) = \sqrt{(\xi_1-\eta_1)^2+(\xi_2-\eta_2)^2+(\xi_3-\eta_3)^2+...+(\xi_n-\eta_n)^2} \tag{3.1.6}$$

where $x=(\xi_1,\xi_2,\xi_3,...,\xi_n), y=(\eta_1,\eta_2,\eta_3,...,\eta_n)\in\mathbb{R}^n$.

Example 3.1.30. *The sequence space* l_∞ *is complete.*

Example 3.1.31. *The sequence space* l_p *is complete.*

Definition 3.1.32. (Inner product space) An inner product space is a vector space X with an inner product defined on X. Here, an inner product on X is a mapping of $X\times X$ into the scalar field $\mathbb{K}$ of X; that is, with every pair of vectors x and y there is associated a scalar which is written by $\langle x,y\rangle$ and called the *inner product* of x and y, such that for all vectors x, y, z and scalars α we have

(IP1) $\langle x,x\rangle \geq 0$;

(IP2) $\langle x,x\rangle = 0 \Leftrightarrow x=0$;

(IP3) $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle$;

(IP4) $\langle x,y\rangle = \overline{\langle y,x\rangle}$;

(IP5) $\langle x+y,z\rangle = \langle x,z\rangle + \langle z,y\rangle$.

Example 3.1.33. *The function* $\langle\cdot,\cdot\rangle : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ *defined by*

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i y_i \text{ for all } x=(x_1,x_2,...,x_n),\ y=(y_1,y_2,...,y_n)\in\mathbb{R}^n \tag{3.1.7}$$

is an inner product on $\mathbb{R}^n$. *In this case* $\mathbb{R}^n$ *with this inner product is called real Euclidean n - space.*

Example 3.1.34. *Let* $\mathbb{C}^n$ *be the set of n - tuples of complex numbers. Then the function* $\langle\cdot,\cdot\rangle : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$ *defined by*

$$\langle x,y\rangle = \sum_{i=1}^{n} x_i \overline{y_i} \text{ for all } x=(x_1,x_2,...,x_n),\ y=(y_1,y_2,...,y_n)\in\mathbb{C}^n \tag{3.1.8}$$

is an inner product on $\mathbb{C}^n$. *In this case* $\mathbb{C}^n$ *with this inner product is called complex Euclidean n - space.*

Example 3.1.35. *Let* l_2 *be the set of all sequences of complex numbers* $(a_1, a_2, \ldots, a_i, \ldots)$ *with* $\sum_{i=1}^{\infty} |a_i|^2 < \infty$. *Then the function* $\langle \cdot, \cdot \rangle : l_2 \times l_2 \to \mathbb{C}$ *defined by*

$$\langle x, y \rangle = \sum_{i=1}^{\infty} x_i \overline{y_i} \text{ for all } x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty} \in l_2 \tag{3.1.9}$$

is an inner product on l_2.

Definition 3.1.36. (Hilbert space) An inner product space which is complete with respect to the induced norm is called a *Hilbert space*.

Example 3.1.37. *The Euclidean space* $\mathbb{R}^n$ *is a Hilbert space with inner product defined by*

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 + \ldots + x_n y_n$$

where $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$.

Example 3.1.38. *The space* l_2 *is a Hilbert space with inner product defined by*

$$\langle x, y \rangle = \sum_{j=1}^{\infty} x_j \overline{y_j},$$

where $x, y \in l_2$.

Proposition 3.1.39. (The Cauchy - Schwarz inequality) *Let* X *be an inner product space. Then the following holds:*

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle \text{ for all } x, y \in X, \tag{3.1.10}$$

i.e.,

$$|\langle x, y \rangle| \leq \|x\| \|y\| \text{ for all } x, y \in X. \tag{3.1.11}$$

Definition 3.1.40. (Bounded linear operator) Let X and Y be normed spaces and $T : X \to Y$ be a linear operator. The operator T is said to be *bounded* if there is a real number $c > 0$ such that for all $x \in X$,

$$\|Tx\| \leq c\|x\|. \tag{3.1.12}$$

Definition 3.1.41. (Level set of convex function) Let $f : H \to \mathbb{R}$ be a convex function with the domain H. Then, for any $\lambda \in \mathbb{R}$, the set

$$V_\lambda = \{x \in H | f(x) \leq \lambda\} \tag{3.1.13}$$

Definition 3.1.42. A sequence $\{x_n\}$ in a Hilbert space H is said to converge weakly to a point x in H if

$$\langle x_n, y\rangle \to \langle x, y\rangle \tag{3.1.14}$$

for all $y \in H$ and denote that $x_n \rightharpoonup x$.

Let H be a real Hilbert space and C be a nonempty subset of H. Recall that a mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C. \tag{3.1.15}$$

A mapping $T : C \to C$ is said to be firmly nonexpansive if, for all $x, y \in C$,

$$\langle x - y, Tx - Ty\rangle \geq \|Tx - Ty\|^2. \tag{3.1.16}$$

F is said to be monotone on C if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \ \forall x, y \in C \tag{3.1.17}$$

F is said to be τ_n - inverse strongly monotone (shortly, τ_n - ism) with $\tau_n > 0$ if

$$\langle F(x) - F(y), x - y\rangle \geq \tau_n \|F(x) - F(y)\|^2, \ \forall x, y \in C; \tag{3.1.18}$$

F is said to be Lipschitz continuous on C with constant $\lambda > 0$ if

$$\|F(x) - F(y)\| \leq \lambda \|x - y\|, \ \forall x, y \in C. \tag{3.1.19}$$

A mapping $f : C \to C$ is said to be a contraction if there exists a constant $a \in [0, 1)$

such that

$$\|f(x) - f(y)\| \leq a\|x - y\|, \ \forall x, y \in C. \tag{3.1.20}$$

A differentiable function f is convex if and only if there holds the inequality:

$$f(z) \geq f(x) + \langle \nabla f(x), z - x \rangle, \ \forall z \in H. \tag{3.1.21}$$

Recall that an element $g \in H$ is said to be a subgradient of $f : H \to \mathbb{R}$ at x if

$$f(z) \geq f(x) + \langle g, z - x \rangle, \ \forall z \in H. \tag{3.1.22}$$

This relation is called the subdifferentiable inequality.

A function $f : H \to \mathbb{R}$ is said to be subdifferentiable at x, if it has at least one subgradient at x.

The set of subgradients of f at the point x is called the subdifferentiable of f at x, and it is denoted by $\partial f(x)$, that is

$$\partial f(x) = \{g \in H | f(z) \geq f(x) + \langle g, z - x \rangle, \forall z \in H\}. \tag{3.1.23}$$

A function f is called subdifferentiable, if it is subdifferentiable at all $x \in H$. If a function f is differentiable and convex, then its gradient and subgradient coincide.

A function $f : H \to \mathbb{R}$ is said to be weakly lower semi-continuous (w - lsc) at x if $x_n \rightharpoonup x$ implies

$$f(x) \leq \liminf_{n \to \infty} f(x_n). \tag{3.1.24}$$

We know that the orthogonal projection P_C from H onto a nonempty closed convex subset $C \subset H$ is a typical example of a firmly nonexpansive mapping, which is defined by

$$P_C x := \arg\min_{y \in C} \|x - y\|^2, \ x \in H. \tag{3.1.25}$$

Lemma 3.1.43. [3] *For any $x \in H$ and $z \in C$, then $z = P_C x$ if and only if*

$$\langle x - z, y - z \rangle \leq 0, \forall y \in C. \tag{3.1.26}$$

Example 3.1.44. *[7] A half - space in a Hilbert space H has the form*

$$H_{(a,\beta)} = \{z \in H : \langle a, z \rangle \leq \beta\}, \tag{3.1.27}$$

where $a \in H$, $a \neq 0$ and $\beta \in \mathbb{R}$. It is clear that $H_{(a,\beta)}$ is closed and convex. We know that

$$P_{H_{(a,\beta)}} x = \begin{cases} x - \frac{\langle a, x \rangle - \beta}{\|a\|^2} a & \text{if } \langle a, x \rangle > \beta \\ x & \text{if } \langle a, x \rangle \leq \beta. \end{cases} \tag{3.1.28}$$

Lemma 3.1.45. *[5] Let C and Q be closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively and $A : H_1 \to H_2$ a bounded linear operator. Let $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ then ∇f is $\|A\|^2$ - Lipschitz continuous.*

Lemma 3.1.46. *[3] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Then for any $x \in H$, the following assertions hold:*

(i) $\langle x - P_C x, z - P_C x \rangle \leq 0$ for all $z \in C$;

(ii) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle$ for all $x, y \in H$;

(iii) $\|P_C x - z\|^2 \leq \|x - z\|^2 - \|P_C x - x\|^2$ for all $z \in C$.

From Lemma 3.1.46, the operator $I - P_C$ is also firmly nonexpansive, where I denotes the identity operator, *i.e.*, for any $x, y \in H$,

$$\langle (I - P_C)x - (I - P_C)y, x - y \rangle \geq \|(I - P_C)x - (I - P_C)y\|^2. \tag{3.1.29}$$

Lemma 3.1.47. *[2] Let S be a nonempty, closed and convex subset of a real Hilbert space H and $\{x_n\}$ be a sequence in H that satisfies the following properties:*

(i) $\lim_{n\to\infty} \|x_n - x\|$ exists for each $x \in S$;

(ii) $\omega_w(x_n) \subset S$.

Then $\{x_n\}$ converges weakly to a point in S.

Lemma 3.1.48. *[20] Assume* $\{s_n\}$ *is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-c_n)s_n + c_n\delta_n, n \geq 1, \tag{3.1.30}$$

$$s_{n+1} \leq s_n - \lambda_n + \varphi_n, n \geq 1, \tag{3.1.31}$$

where $\{c_n\}$ *is a sequence in* $(0,1)$, $\{\varphi_n\}$ *is a sequence of nonnegative real numbers and* $\{\delta_n\}$ *and* $\{\varphi_n\}$ *are two sequences in* $\mathbb{R}$ *such that*

(i) $\sum_{n=1}^{\infty} c_n = \infty$*;*

(ii) $\lim_{n\to\infty} \varphi_n = 0$*;*

(iii) $\lim_{k\to\infty} \lambda_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \delta_{n_k} \leq 0$ *for any subsequence* $\{n_k\}$ *of* $\{n\}$*.*

Then $\lim_{n\to\infty} s_n = 0$*.*

CHAPTER IV

RESULTS

4.1 Strong convergence of the modified projection and contraction methods

In this section, we introduce a projection algorithm using linesearch for the strong convergence theorem. Let H_1 and H_2 be are real Hilbert spaces and C, Q be closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator and A^* is the adjoint operator of A. We define $F : H_1 \to H_1$ by

$$F(x) = A^*(I - P_Q)A(x). \tag{4.1.1}$$

Algorithm 4.1.1. *Let $f : H_1 \to H_1$ be a contraction. For any $\sigma > 0, \rho \in (0,1)$ and $\mu \in (0,1)$, choose an arbitrary initial guess $x_1 \in H_1$. Assume x_n and y_n have been constructed. Compute the sequence x_{n+1} via the formula*

$$y_n = P_C(x_n - \tau_n F(x_n)) \tag{4.1.2}$$

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)(x_n - \gamma \phi_n d(x_n, y_n)) \tag{4.1.3}$$

where $\{\alpha_n\} \subseteq (0,1)$, $\gamma \in (0,2)$ and $\tau_n = \sigma \rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\tau_n \|F(x_n) - F(y_n)\| \leq \mu \|x_n - y_n\|, \tag{4.1.4}$$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n (F(x_n) - F(y_n)) \tag{4.1.5}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n \|(I - P_Q)Ay_n\|^2}{\|d(x_n, y_n)\|^2}. \tag{4.1.6}$$

Lemma 4.1.2. *[50] The line rule (4.1.4) is well defined. Besides, $\tau' \leq \tau_n \leq \sigma$, where $\tau' = \min\{\sigma, \frac{\mu\rho}{L}\}$.*

Lemma 4.1.3. *[14] Let $\{x_n\}$ and $\{y_n\}$ be the iterations generated by Algorithm 4.1.1.*

Then we have

$$\langle x_n - z, d(x_n, y_n)\rangle \geq \phi_n \|d(x_n, y_n)\|^2, \ \forall z \in S. \tag{4.1.7}$$

Lemma 4.1.4. *[14] Let $\{x_n\}$ and $\{y_n\}$ be the iterations generated by Algorithm 4.1.1. Then we have*

$$\langle x_n - y_n, d(x_n, y_n)\rangle \geq (1-\mu)\|x_n - y_n\|^2 \tag{4.1.8}$$

and

$$\phi_n \geq \frac{1-\mu}{1+\mu^2}. \tag{4.1.9}$$

Theorem 4.1.5. *Assume that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $S \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 4.1.1 converges strongly to $z = P_S f(z)$ in S.*

Proof. We set $z = P_S f(z)$. Then, by Lemma 4.1.3, we have

$$\begin{aligned}
\|x_n - \gamma\phi_n d(x_n, y_n) - z\|^2 &= \|x_n - z\|^2 - 2\gamma\phi_n\langle x_n - z, d(x_n, y_n)\rangle \\
&\quad +\gamma^2\phi_n^2\|d(x_n, y_n)\|^2 \\
&\leq \|x_n - z\|^2 - 2\gamma\phi_n^2\|d(x_n, y_n)\|^2 + \gamma^2\phi_n^2\|d(x_n, y_n)\|^2 \\
&= \|x_n - z\|^2 - \gamma(2-\gamma)\phi_n^2\|d(x_n, y_n)\|^2. \qquad (4.1.10)
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) - z\|^2 \\
&= \langle \alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) - z, x_{n+1} - z\rangle \\
&= \alpha_n\langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +(1-\alpha_n)\langle x_n - \gamma\phi_n d(x_n, y_n) - z, x_{n+1} - z\rangle \\
&\leq \alpha_n\|f(x_n) - f(z)\|\|x_{n+1} - z\| + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +(1-\alpha_n)\|x_n - \gamma\phi_n d(x_n, y_n) - z\|\|x_{n+1} - z\| \\
&\leq \frac{1}{2}\alpha_n(\|f(x_n) - f(z)\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad +\frac{1}{2}(1-\alpha_n)(\|x_n - \gamma\phi_n d(x_n, y_n) - z\|^2 + \|x_{n+1} - z\|^2)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\alpha_n a\|x_n - z\|^2 + \frac{1}{2}\alpha_n\|x_{n+1} - z\|^2 + \alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&+\frac{1}{2}(1-\alpha_n)(\|x_n - z\|^2 - \gamma(2-\gamma)\phi_n^2\|d(x_n, y_n)\|^2 \\
&+\|x_{n+1} - z\|^2). \qquad (4.1.11)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - z\|^2 &\leq (1 - \alpha_n(1-a))\|x_n - z\|^2 + 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
&\quad -(1-\alpha_n)\gamma(2-\gamma)\phi_n^2\|d(x_n, y_n)\|^2. \qquad (4.1.12)
\end{aligned}
$$

Next, we will show that $\{x_n\}$ is bounded. We see that

$$
\begin{aligned}
\|x_{n+1} - z\| &= \|\alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) - z\| \\
&\leq \alpha_n\|f(x_n) - z\| + (1-\alpha_n)\|x_n - \gamma\phi_n d(x_n, y_n) - z\| \\
&\leq \alpha_n\|f(x_n) - f(z)\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&\leq \alpha_n a\|x_n - z\| + \alpha_n\|f(z) - z\| + (1-\alpha_n)\|x_n - z\| \\
&= (1 - \alpha_n(1-a))\|x_n - z\| + \alpha_n\|f(z) - z\|. \qquad (4.1.13)
\end{aligned}
$$

By induction, we can show that $\{x_n\}$ is bounded. Employing Lemma 3.1.48 and (4.1.12), we set

$$
\begin{aligned}
s_n &= \|x_n - z\|^2 \\
\varphi_n &= 2\alpha_n\langle f(z) - z, x_{n+1} - z\rangle \\
\delta_n &= \frac{2}{1-a}\langle f(z) - z, x_{n+1} - z\rangle \\
\lambda_n &= (1-\alpha_n)\gamma(2-\gamma)\phi_n^2\|d(x_n, y_n)\|^2 \\
c_n &= (1-a)\alpha_n. \qquad (4.1.14)
\end{aligned}
$$

So, (4.1.12) reduces to the inequalities

$$
s_{n+1} \leq (1-c_n)s_n + c_n\delta_n \qquad (4.1.15)
$$

$$
s_{n+1} \leq s_n - \lambda_n + \varphi_n. \qquad (4.1.16)
$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\lim_{k\to\infty} \lambda_{n_k} = 0. \tag{4.1.17}$$

It follows that

$$\lim_{k\to\infty} (1-\alpha_{n_k})\gamma(2-\gamma)\phi_{n_k}^2 \|d(x_{n_k}, y_{n_k})\|^2 = 0. \tag{4.1.18}$$

Using Lemma 4.1.4, we obtain

$$\lim_{k\to\infty} \|d(x_{n_k}, y_{n_k})\| = 0. \tag{4.1.19}$$

By (4.1.5) we see that

$$\begin{aligned} \|x_{n_k} - y_{n_k}\| &\le \|d(x_{n_k}, y_{n_k})\| + \tau_{n_k}\|F(x_{n_k}) - F(y_{n_k})\| \\ &\le \|d(x_{n_k}, y_{n_k})\| + \mu\|x_{n_k} - y_{n_k}\|. \end{aligned} \tag{4.1.20}$$

It follows that

$$(1-\mu)\|x_{n_k} - y_{n_k}\| \le \|d(x_{n_k}, y_{n_k})\|. \tag{4.1.21}$$

From (4.1.19), we obtain

$$\lim_{k\to\infty} \|x_{n_k} - y_{n_k}\| = 0. \tag{4.1.22}$$

Consider

$$\begin{aligned} \|x_{n_k+1} - x_{n_k}\| &= \|\alpha_{n_k} f(x_{n_k}) + (1-\alpha_{n_k})(x_{n_k} - \gamma\phi_{n_k} d(x_{n_k}, y_{n_k})) - x_{n_k}\| \\ &\le \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\|x_{n_k} - \gamma\phi_{n_k} d(x_{n_k}, y_{n_k}) - x_{n_k}\| \\ &= \alpha_{n_k}\|f(x_{n_k}) - x_{n_k}\| + (1-\alpha_{n_k})\gamma\phi_{n_k}\|d(x_{n_k}, y_{n_k})\| \\ &\to 0, \text{ as } k\to\infty. \end{aligned} \tag{4.1.23}$$

By the definitions of $\{y_{n_k}\}$ and $d(x_{n_k}, y_{n_k})$, we get

$$y_{n_k} = P_C(y_{n_k} - (\tau_{n_k} F(y_{n_k}) - d(x_{n_k}, y_{n_k}))). \tag{4.1.24}$$

From Lemma 3.1.43, it follows that

$$\langle x - y_{n_k}, \tau_{n_k} F(y_{n_k}) - d(x_{n_k}, y_{n_k})\rangle \geq 0, \ \forall x \in C. \tag{4.1.25}$$

Take arbitrarily $z \in S \subset C$. By setting $x = z$ in (4.1.25), we have

$$\langle y_{n_k} - z, d(x_{n_k}, y_{n_k}) - \tau_{n_k} F(y_{n_k})\rangle \geq 0, \tag{4.1.26}$$

which implies that

$$\langle y_{n_k} - z, d(x_{n_k}, y_{n_k})\rangle \geq \tau_{n_k}\langle y_{n_k} - z, F(y_{n_k})\rangle. \tag{4.1.27}$$

Since $\{x_{n_k}\}$ is bounded, the set $\omega_w(x_{n_k})$ is nonempty. Let $x^* \in \omega_w(x_{n_k})$. Then there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}} \rightharpoonup x^*$. We also have $Ax_{n_{k_i}} \rightharpoonup Ax^*$. In fact

$$\begin{aligned} |\langle Ax_{n_{k_i}}, z\rangle - \langle Ax^*, z\rangle| &= |\langle Ax_{n_{k_i}} - Ax^*, z\rangle| \\ &= |\langle A(x_{n_{k_i}} - x^*), z\rangle| \\ &= |\langle x_{n_{k_i}} - x^*, A^* z\rangle| \\ &\to 0. \end{aligned} \tag{4.1.28}$$

Next, we show that x^* is a solution of the SFP. From (4.1.19) and the boundedness of $\{y_{n_k}\}$, we have

$$\begin{aligned} \tau_{n_k}\|Ay_{n_k} - P_Q Ay_{n_k}\|^2 &\leq \tau_{n_k}\langle Ay_{n_k} - Az, (I - P_Q)Ay_{n_k} - (I - P_Q)Az\rangle \\ &= \tau_{n_k}\langle Ay_{n_k} - Az, (I - P_Q)Ay_{n_k}\rangle \\ &= \tau_{n_k}\langle y_{n_k} - z, A^T(I - P_Q)Ay_{n_k}\rangle \\ &= \tau_{n_k}\langle y_{n_k} - z, F(y_{n_k})\rangle. \end{aligned} \tag{4.1.29}$$

By (4.1.27), (4.1.19) and Lemma 3.1.46, we have

$$\begin{aligned} \|Ay_{n_k} - P_Q Ay_{n_k}\|^2 &\leq \frac{1}{\tau'}\langle y_{n_k} - z, d(x_{n_k}, y_{n_k})\rangle \\ &\leq \frac{1}{\tau'}\|y_{n_k} - z\|\|d(x_{n_k}, y_{n_k})\| \\ &\to 0. \end{aligned} \tag{4.1.30}$$

Hence,

$$\lim_{k\to\infty} \|Ay_{n_k} - P_Q Ay_{n_k}\| = 0. \tag{4.1.31}$$

Thus $Ax^* \in Q$. From (4.1.1) and (4.1.31), it follows that $\lim_{k\to\infty} \|F(y_{n_k})\| = 0$. By (4.1.2) and Lemma 3.1.46 (iii), we have

$$\begin{aligned} \|y_{n_k} - P_C(y_{n_k})\| &\leq \|x_{n_k} - y_{n_k} - \tau_{n_k} F(x_{n_k})\| \\ &\leq \|x_{n_k} - y_{n_k}\| + \tau_{n_k}\|F(x_{n_k})\| \\ &\leq \|x_{n_k} - y_{n_k}\| + \tau_{n_k}\|F(x_{n_k}) - F(y_{n_k})\| + \tau_{n_k}\|F(y_{n_k})\| \\ &\leq (1+\mu)\|x_{n_k} - y_{n_k}\| + \tau_{n_k}\|F(y_{n_k})\| \\ &\to 0 \text{ as } k \to \infty, \end{aligned} \tag{4.1.32}$$

which implies $x^* \in C$. From Lemma 3.1.46 (i), we obtain

$$\begin{aligned} \limsup_{k\to\infty}\langle f(z) - z, x_{n_k} - z\rangle &= \lim_{i\to\infty}\langle f(z) - z, x_{n_{k_i}} - z\rangle \\ &= \langle f(z) - z, x^* - z\rangle \\ &\leq 0. \end{aligned} \tag{4.1.33}$$

From (4.1.23) and (4.1.33), we obtain

$$\limsup_{k\to\infty}\langle f(z) - z, x_{n_k+1} - z\rangle \leq 0. \tag{4.1.34}$$

Hence, we get

$$\limsup_{k\to\infty} \delta_{n_k} \leq 0. \tag{4.1.35}$$

Using Lemma 3.1.48, we conclude that the sequence $\{x_n\}$ converges strongly to $z = P_S f(z)$. □

Next, we introduce the modified relaxation projection and contraction methods, in which the closed convex subsets C and Q have particular structure.

By the definition of the subgradient, it is clear that $C \subseteq C_n$ and $Q \subseteq Q_n$. The projections onto C_n and Q_n are easy to compute since C_n and Q_n are two half - spaces.

Define $F_n : H_1 \to H_1$ by

$$F_n(x) = A^*(I - P_{Q_n})A(x). \tag{4.1.36}$$

Algorithm 4.1.6. *Let $f : H_1 \to H_1$ be a contraction. For any $\sigma > 0, \rho \in (0,1)$ and $\mu \in (0,1)$, choose an arbitrary initial guess $x_1 \in H_1$. Assume x_n and y_n have been constructed. Compute the sequence x_{n+1} via the formula*

$$\begin{aligned} y_n &= P_{C_n}(x_n - \tau_n F_n(x_n)) \\ x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) \end{aligned} \tag{4.1.37}$$

where $\{\alpha_n\} \subseteq (0,1)$, $\gamma \in (0,2)$ and $\tau_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\tau_n\|F_n(x_n) - F_n(y_n)\| \leq \mu\|x_n - y_n\|, \tag{4.1.38}$$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n(F_n(x_n) - F_n(y_n)) \tag{4.1.39}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n\|(I - P_{Q_n})Ay_n\|^2}{\|d(x_n, y_n)\|^2}. \tag{4.1.40}$$

Theorem 4.1.7. *Assume that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $S \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 4.1.6 converges strongly to $z = P_S f(z)$ in S.*

Proof. As in the proof of Theorem 4.1.5, we see that

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le (1-\alpha_n(1-a))\|x_n-z\|^2+2\alpha_n\langle f(z)-z, x_{n+1}-z\rangle \\ &\quad -(1-\alpha_n)\gamma(2-\gamma)\phi_n^2\|d(x_n,y_n)\|^2. \qquad (4.1.41)\end{aligned}$$

Moreover, the sequence $\{x_n\}$ is bounded and $\|x_n - y_n\| \to 0$. Let x^* be a cluster point of $\{x_n\}$ with $\{x_{n_k}\}$ converging to x^*. From (4.1.22), it follows that $\{y_{n_k}\}$ also converges to x^*.

Now, we show that x^* is a solution of the SFP. In fact, since $y_{n_k} \in C_{n_k}$, by the definition of $\{C_{n_k}\}$, we have

$$c(x_{n_k}) + \langle \xi_{n_k}, y_{n_k} - x_{n_k}\rangle \le 0, \qquad (4.1.42)$$

where $\xi_{n_k} \in \partial c(x_{n_k})$. Since ∂c is bounded and (4.1.22), we have

$$\begin{aligned}c(x_{n_k}) &\le \langle \xi_{n_k}, x_{n_k} - y_{n_k}\rangle \\ &\le \|\xi_{n_k}\|\|y_{n_k} - x_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \qquad (4.1.43)\end{aligned}$$

which implies $c(x^*) \le 0$, i.e., $x^* \in C$. As in Theorem 4.1.5, we can show that $\|A_{y_{n_k}} - P_{Q_{n_k}}(A_{y_{n_k}})\| \to 0$ as $k \to \infty$. Since $P_{Q_{n_k}}(Ay_{n_k}) \in Q_{n_k}$, we have

$$q(Ay_{n_k}) + \langle \eta_{n_k}, P_{Q_{n_k}}(Ay_{n_k}) - Ay_{n_k}\rangle \le 0 \qquad (4.1.44)$$

where $\eta_{n_k} \in \partial q(Ay_{n_k})$. From (4.1.31), we obtain

$$\begin{aligned}q(Ay_{n_k}) &\le \|\eta_{n_k}\|\|P_{Q_{n_k}}(Ay_{n_k}) - Ay_{n_k}\| \\ &\to 0 \text{ as } k \to \infty. \qquad (4.1.45)\end{aligned}$$

Similarly, we have $q(Ax^*) \le 0$, i.e., $Ax^* \in Q$. Thus x^* is a solution of the SFP.

Following the line of the proof of Theorem 4.1.5 we get that $\{x_n\}$ converges strongly to $z = P_S f(z)$. □

4.2 On the convergence analysis of the gradient - CQ algorithms for the split feasibility problems

In this section, we propose a new gradient - CQ algorithm and derive the weak convergence theorem of the generated sequences by the proposed method.

Define $f_n : H_1 \to \mathbb{R}$ by $f_n(x) = \frac{1}{2}\|(I - P_{Q_n})Ax\|^2$.

Algorithm 4.2.1. *Choose an arbitrary initial guess x_1. Assume x_n and y_n have been constructed. Compute x_{n+1} via the formula*

$$y_n = x_n - \tau_n F_n(x_n) \tag{4.2.1}$$

$$x_{n+1} = P_{C_n}(y_n - \varphi_n F_n(y_n)) \tag{4.2.2}$$

where F_n is defined by 4.1.36 and C_n, Q_n are given as (2.1.4), (2.1.5)

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|F_n(x_n)\|^2 + \theta_n} \text{ and } \varphi_n = \frac{\rho_n f_n(y_n)}{\|F_n(y_n)\|^2 + \theta_n},\ 0 < \rho_n < 4, 0 < \theta_n < 1. \tag{4.2.3}$$

Remark 4.2.2. We see that the iterate y_n is defined by a gradient method with the stepsize τ_n and the iterate x_{n+1} is defined by a relaxed CQ algorithm with the stepsize φ_n. Here a parameter θ_n is added to ensure that the sequence $\{x_n\}$ generated by our algorithm has an infinite number of iterations.

We next state our weak convergence theorem.

Theorem 4.2.3. *Assume that $\inf_n \rho_n(4 - \rho_n) > 0$ and $\lim_{n\to\infty} \theta_n = 0$. Then the sequence $\{x_n\}$ generated by Algorithm 4.2.1 converges weakly to a point of S.*

Proof. Let $z \in S$. Since $C \subseteq C_n$ and $Q \subseteq Q_n$, we have $z = P_C(z) = P_{C_n}(z)$ and $Az = P_Q(Az) = P_{Q_n}(Az)$. It follows that $F_n(z) = 0$. Using Lemma 3.1.46 (iii), we see that

$$\begin{aligned}\|x_{n+1}-z\|^2 &= \|P_{C_n}(y_n-\varphi_n F_n(y_n))-z\|^2\\ &\le \|y_n-\varphi_n F_n(y_n)-z\|^2-\|x_{n+1}-y_n+\varphi_n F_n(y_n)\|^2\\ &= \|y_n-z\|^2+\varphi_n^2\|F_n(y_n)\|^2-2\varphi_n\langle y_n-z,F_n(y_n)\rangle\\ &\quad -\|x_{n+1}-y_n+\varphi_n F_n(y_n)\|^2. \end{aligned} \tag{4.2.4}$$

From (3.1.29) and $F_n(z)=0$, we obtain

$$\begin{aligned}\langle y_n-z,F_n(y_n)\rangle &= \langle y_n-z,F_n(y_n)-F_n(z)\rangle\\ &= \langle y_n-z,A^*(I-P_{Q_n})Ay_n-A^*(I-P_{Q_n})Az\rangle\\ &= \langle Ay_n-Az,(I-P_{Q_n})Ay_n-(I-P_{Q_n})Az\rangle\\ &\ge \|(I-P_{Q_n})Ay_n\|^2\\ &= 2f_n(y_n). \end{aligned} \tag{4.2.5}$$

It also follows that

$$\langle x_n-z,F_n(x_n)\rangle \ge 2f_n(x_n). \tag{4.2.6}$$

Moreover, by (4.2.6), we see that

$$\begin{aligned}\|y_n-z\|^2 &= \|x_n-\tau_n F_n(x_n)-z\|^2\\ &= \|x_n-z\|^2+\tau_n^2\|F_n(x_n)\|^2-2\tau_n\langle x_n-z,F_n(x_n)\rangle\\ &\le \|x_n-z\|^2+\tau_n^2\|F_n(x_n)\|^2-4\tau_n f_n(x_n). \end{aligned} \tag{4.2.7}$$

Combining (4.2.4)-(4.2.7), we obtain

$$\begin{aligned}\|x_{n+1}-z\|^2 &\le \|x_n-z\|^2+\tau_n^2\|F_n(x_n)\|^2-4\tau_n f_n(x_n)+\varphi_n^2\|F_n(y_n)\|^2\\ &\quad -4\varphi_n f_n(y_n)-\|x_{n+1}-y_n+\varphi_n F_n(y_n)\|^2\\ &= \|x_n-z\|^2+\frac{\rho_n^2 f_n^2(x_n)}{(\|F_n(x_n)\|^2+\theta_n)^2}\|F_n(x_n)\|^2\\ &\quad -\frac{4\rho_n f_n^2(x_n)}{\|F_n(x_n)\|^2+\theta_n}+\frac{\rho_n^2 f_n^2(y_n)}{(\|F_n(y_n)\|^2+\theta_n)^2}\|F_n(y_n)\|^2 \end{aligned}$$

$$
\begin{aligned}
& -\frac{4\rho_n f_n^2(y_n)}{\|F_n(y_n)\|^2+\theta_n} - \|x_{n+1} - y_n + \varphi_n F_n(y_n)\|^2 \\
\leq \ & \|x_n - z\|^2 + \frac{\rho_n^2 f_n^2(x_n)}{\|F_n(x_n)\|^2+\theta_n} - \frac{4\rho_n f_n^2(x_n)}{\|F_n(x_n)\|^2+\theta_n} \\
& + \frac{\rho_n^2 f_n^2(y_n)}{\|F_n(y_n)\|^2+\theta_n} - \frac{4\rho_n f_n^2(y_n)}{\|F_n(y_n)\|^2+\theta_n} \\
& - \|x_{n+1} - y_n + \varphi_n F_n(y_n)\|^2 \\
= \ & \|x_n - z\|^2 - \rho_n(4-\rho_n)\frac{f_n^2(x_n)}{\|F_n(x_n)\|^2+\theta_n} \qquad (4.2.8) \\
& -\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|F_n(y_n)\|^2+\theta_n} - \|x_{n+1} - y_n + \varphi_n F_n(y_n)\|^2.
\end{aligned}
$$

This implies that, since $0 < \rho_n < 4$,

$$
\|x_{n+1} - z\| \leq \|x_n - z\|. \qquad (4.2.9)
$$

Thus $\lim_{n\to\infty} \|x_n - z\|$ exists and hence $\{x_n\}$ is bounded. Again, from (4.2.8), it follows that

$$
\liminf_{n\to\infty} \rho_n(4-\rho_n)\frac{f_n^2(x_n)}{\|F_n(x_n)\|^2+\theta_n} = 0, \qquad (4.2.10)
$$

which implies by our assumptions that

$$
\lim_{n\to\infty} \frac{f_n^2(x_n)}{\|F_n(x_n)\|^2} = 0. \qquad (4.2.11)
$$

We can check that $\{\|F_n(x_n)\|\}$ is bounded by Lemma 3.1.45. So we get $\lim_{n\to\infty} f_n(x_n) = 0$. This means $\lim_{n\to\infty} \|(I - P_{Q_n})Ax_n\| = 0$. Also, we have $\lim_{n\to\infty} f_n(y_n) = \lim_{n\to\infty} \|(I - P_{Q_n})Ay_n\| = 0$.

Furthermore, from (4.2.8) we obtain

$$
\lim_{n\to\infty} \|x_{n+1} - y_n + \varphi_n F_n(y_n)\| = 0. \qquad (4.2.12)
$$

We note that

$$
\varphi_n\|F_n(y_n)\| = \frac{\rho_n f_n(y_n)}{\|F_n(y_n)\|^2+\theta_n}\|F_n(y_n)\| \to 0, \text{ as } n \to \infty. \qquad (4.2.13)
$$

Hence, by (4.2.12) and (4.2.13) we obtain $\lim_{n\to\infty} \|x_{n+1} - y_n\| = 0$. On the otherhand, from (4.2.1) and $\tau_n \|F_n(x_n)\| \to 0$ as $n \to \infty$ we get $\lim_{n\to\infty} \|y_n - x_n\| = 0$. Hence $\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0$.

Since $\{x_n\}$ is bounded, the set $\omega_w(x_n)$ is nonempty. Let $x^* \in \omega_w(x_n)$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup x^* \in H_1$.

Next, we show that x^* is in S. Since $x_{n_k+1} \in C_{n_k}$, by the definition of C_{n_k}, we get

$$c(x_{n_k}) \leq \langle \xi_{n_k}, x_{n_k} - x_{n_k+1} \rangle \tag{4.2.14}$$

where $\xi_{n_k} \in \partial c(x_{n_k})$. It follows that, by the boundedness of ∂c,

$$\begin{aligned} c(x_{n_k}) &\leq \|\xi_{n_k}\| \|x_{n_k} - x_{n_k+1}\| \\ &\to 0, \text{ as } k \to \infty. \end{aligned} \tag{4.2.15}$$

By the w-lsc of c, $x_{n_k} \rightharpoonup x^*$ and (4.2.15), we conclude that

$$c(x^*) \leq \liminf_{k\to\infty} c(x_{n_k}) \leq 0. \tag{4.2.16}$$

Thus $x^* \in C$.

Next, we prove that $Ax^* \in Q$. Since $P_{Q_{n_k}}(Ax_{n_k}) \in Q_{n_k}$, we have

$$q(Ax_{n_k}) \leq \langle \eta_{n_k}, Ax_{n_k} - P_{Q_{n_k}}(Ax_{n_k}) \rangle \tag{4.2.17}$$

where $\eta_{n_k} \in \partial q(Ax_{n_k})$. So we obtain

$$\begin{aligned} q(Ax_{n_k}) &\leq \|\eta_{n_k}\| \|Ax_{n_k} - P_{Q_{n_k}}(Ax_{n_k})\| \\ &\to 0, \text{ as } k \to \infty. \end{aligned} \tag{4.2.18}$$

The w - lsc of q and (4.2.18) imply that

$$q(Ax^*) \leq \liminf_{k\to\infty} q(Ax_{n_k}) \leq 0. \tag{4.2.19}$$

Thus, $Ax^* \in Q$. Using Lemma 3.1.47, we conclude that the sequence $\{x_n\}$ converges weakly to a point in S. $\square$

Next, we present another gradient - CQ algorithm by using the Halpern iteration for the strong convergence theorem.

Algorithm 4.2.4. *Choose an arbitrary initial guess x_1. Assume x_n and y_n have been constructed. Compute the sequence x_{n+1} via the formula*

$$y_n = x_n - \tau_n F_n(x_n) \tag{4.2.20}$$

$$x_{n+1} = \alpha_n u + (1-\alpha_n) P_{C_n}(y_n - \varphi_n F_n(y_n)) \tag{4.2.21}$$

where $\{\alpha_n\} \subset (0,1)$, $u \in H_1$, C_n and Q_n are given as (2.1.4), (2.1.5)

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|F_n(x_n)\|^2 + \theta_n} \text{ and } \varphi_n = \frac{\rho_n f_n(y_n)}{\|F_n(y_n)\|^2 + \theta_n}, 0 < \rho_n < 4, 0 < \theta_n < 1. \tag{4.2.22}$$

Theorem 4.2.5. *Assume that $\{\alpha_n\}$, $\{\rho_n\}$ and $\{\theta_n\}$ satisfy the assumptions:*

(a1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(a2) $\inf_n \rho_n(4-\rho_n) > 0$;

(a3) $\lim_{n\to\infty} \theta_n = 0$.

Then the sequence $\{x_n\}$ generated by Algorithm 4.2.4 converges strongly to a point $P_S u$.

Proof. We set $z = P_S u$. Using the proof line as in Theorem 4.2.3, we obtain

$$\begin{aligned}\|P_{C_n}(y_n - \varphi_n F_n(y_n)) - z\|^2 &\leq \|y_n - z\|^2 - \rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|F_n(y_n)\|^2 + \theta_n} \\ &\quad -\|P_{C_n}(y_n - \varphi_n F_n(y_n)) - y_n \\ &\quad +\varphi_n F_n(y_n)\|^2 \end{aligned} \tag{4.2.23}$$

and

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 - \rho_n(4-\rho_n)\frac{f_n^2(x_n)}{\|F_n(x_n)\|^2 + \theta_n}. \tag{4.2.24}$$

On the other hand, we have

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|\alpha_n(u-z) + (1-\alpha_n)(P_{C_n}(y_n - \varphi_n F_n(y_n)) - z)\|^2 \\ &\leq (1-\alpha_n)\|P_{C_n}(y_n - \varphi_n F_n(y_n)) - z\|^2 \\ &\quad +2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{4.2.25}$$

Combining (4.2.23) - (4.2.25), we obtain

$$\begin{aligned} \|x_{n+1} - z\|^2 &\leq (1-\alpha_n)\|x_n - z\|^2 - (1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(x_n)}{\|F_n(x_n)\|^2 + \theta_n} \\ &\quad -(1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|F_n(y_n)\|^2 + \theta_n} \\ &\quad -(1-\alpha_n)\|P_{C_n}(y_n - \varphi_n F_n(y_n)) - y_n + \varphi_n F_n(y_n)\|^2 \\ &\quad +2\alpha_n\langle u - z, x_{n+1} - z\rangle. \end{aligned} \tag{4.2.26}$$

Next, we will show that $\{x_n\}$ is bounded. Again, using (4.2.23) and (4.2.24), we get

$$\begin{aligned} \|x_{n+1} - z\| &= \|\alpha_n u + (1-\alpha_n)P_{C_n}(y_n - \varphi_n F_n(y_n)) - z\| \\ &\leq \alpha_n\|u - z\| + (1-\alpha_n)\|y_n - z\| \\ &\leq \alpha_n\|u - z\| + (1-\alpha_n)\|x_n - z\|. \end{aligned} \tag{4.2.27}$$

By induction, we can show that $\{x_n\}$ is bounded. Employing Lemma 3.1.48, from (4.2.26), we set

$$\begin{aligned} s_n &= \|x_n - z\|^2; \\ \gamma_n &= 2\alpha_n\langle u - z, x_{n+1} - z\rangle; \\ \delta_n &= 2\langle u - z, x_{n+1} - z\rangle; \\ \lambda_n &= (1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(x_n)}{\|F_n(x_n)\|^2 + \theta_n} \\ &\quad +(1-\alpha_n)\rho_n(4-\rho_n)\frac{f_n^2(y_n)}{\|F_n(y_n)\|^2 + \theta_n} \end{aligned}$$

$$+(1-\alpha_n)\|P_{C_n}(y_n-\varphi_n F_n(y_n))-y_n+\varphi_n F_n(y_n)\|^2. \qquad (4.2.28)$$

So (4.2.26) reduces to the inequalities

$$s_{n+1} \leq (1-\alpha_n)s_n+\alpha_n\delta_n, n\geq 1 \qquad (4.2.29)$$

$$s_{n+1} \leq s_n-\lambda_n+\gamma_n. \qquad (4.2.30)$$

Let $\{n_k\}$ be a subsequence of $\{n\}$ and suppose that

$$\lim_{k\to\infty}\lambda_{n_k}=0. \qquad (4.2.31)$$

Then we have

$$\begin{aligned}\lim_{k\to\infty}&(1-\alpha_{n_k})\rho_{n_k}(4-\rho_{n_k})\frac{f^2_{n_k}(x_{n_k})}{\|F_{n_k}(x_{n_k})\|^2+\theta_{n_k}}\\ &+(1-\alpha_{n_k})\rho_{n_k}(4-\rho_{n_k})\frac{f^2_{n_k}(y_{n_k})}{\|F_{n_k}(y_{n_k})\|^2+\theta_{n_k}}\\ &+(1-\alpha_{n_k})\|P_{C_{n_k}}(y_{n_k}-\varphi_{n_k}F_{n_k}(y_{n_k}))-y_{n_k}+\varphi_{n_k}F_{n_k}(y_{n_k})\|^2\\ &=0 \qquad (4.2.32)\end{aligned}$$

which implies, by our assumptions

$$\frac{f^2_{n_k}(x_{n_k})}{\|F_{n_k}(x_{n_k})\|^2}\to 0,\ \frac{f^2_{n_k}(y_{n_k})}{\|F_{n_k}(y_{n_k})\|^2}\to 0 \text{ and}$$
$$\|P_{C_{n_k}}(y_{n_k}-\varphi_{n_k}F_{n_k}(y_{n_k}))-y_{n_k}+\varphi_{n_k}F_{n_k}(y_{n_k})\|\to 0$$

as $k\to\infty$. Since $\{\|F_{n_k}(x_{n_k})\|\}$ and $\{\|F_{n_k}(y_{n_k})\|\}$ are bounded, it follows that $f_{n_k}(x_{n_k})\to 0$ and $f_{n_k}(y_{n_k})\to 0$ as $k\to\infty$. So we get $\lim_{k\to\infty}\|(I-P_{Q_{n_k}})Ax_{n_k}\|=0$ and $\lim_{k\to\infty}\|(I-P_{Q_{n_k}})Ay_{n_k}\|=0$.

As the same proof in Theorem 4.2.3, we can show that $\omega_w(x_{n_k})\subset S$. Hence there exists a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_i}}\rightharpoonup x^*\in S$.

From Lemma 3.1.46 (i), we obtain

$$\begin{aligned}\limsup_{k\to\infty}\langle u-z, x_{n_k}-z\rangle &= \lim_{i\to\infty}\langle u-z, x_{n_{k_i}}-z\rangle \\ &= \langle u-z, x^*-z\rangle \\ &\leq 0. \end{aligned} \tag{4.2.33}$$

On the other hand, we see that

$$\begin{aligned} &\|x_{n_k+1}-y_{n_k}\| \\ =\ & \|\alpha_{n_k}u+(1-\alpha_{n_k})P_{C_{n_k}}(y_{n_k}-\varphi_{n_k}F_{n_k}(y_{n_k}))-y_{n_k}\| \\ \leq\ & \alpha_{n_k}\|u-y_{n_k}\|+(1-\alpha_{n_k})\|P_{C_{n_k}}(y_{n_k}-\varphi_{n_k}F_{n_k}(y_{n_k}))-y_{n_k}\| \\ \leq\ & \alpha_{n_k}\|u-y_{n_k}\|+(1-\alpha_{n_k})\|P_{C_{n_k}}(y_{n_k}-\varphi_{n_k}F_{n_k}(y_{n_k}))-y_{n_k}+\varphi_{n_k}F_{n_k}(y_{n_k})\| \\ & +(1-\alpha_{n_k})\varphi_{n_k}\|F_{n_k}(y_{n_k})\| \\ \to\ & 0 \text{ as } k\to\infty. \end{aligned} \tag{4.2.34}$$

So, we have

$$\begin{aligned}\|x_{n_k+1}-x_{n_k}\| &\leq \|x_{n_k+1}-y_{n_k}\|+\|y_{n_k}-x_{n_k}\| \\ &\to 0 \text{ as } k\to\infty. \end{aligned} \tag{4.2.35}$$

From (4.2.33) and (4.2.35) we obtain

$$\limsup_{k\to\infty}\langle u-z, x_{n_k+1}-z\rangle \leq 0. \tag{4.2.36}$$

Hence, we get

$$\limsup_{k\to\infty}\delta_{n_k}\leq 0. \tag{4.2.37}$$

Using Lemma 3.1.48, we conclude that the sequence $\{x_n\}$ converges strongly to $z = P_S u$. □

4.3 Numerical examples and Applications

In this section, we present some numerical examples and illustrate its performance by using Algorithm 4.1.1 in Theorem 4.1.5 and Algorithm 4.1.6 in Theorem 4.1.7.

Example 4.3.1. *Let*

$$C = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \|(x_1, x_2, x_3) - (0.5, 0, 0)\|_2 \leq 10\},$$
$$Q = \{(y_1, y_2, y_3) \in \mathbb{R}^3 : 15 \leq y_1 \leq 25, y_2 = 0, y_3 = 0\},$$

and $A = \begin{pmatrix} -1 & 0 & -9 \\ 5 & 9 & 1 \\ -1 & 0 & 1 \end{pmatrix}$. *Choose* $\alpha_n = \frac{1}{100n}$, *for all* $n \in \mathbb{N}$ *and* $f(x) = \frac{1}{2}x$ *where* $x \in \mathbb{R}^3$. *The stopping criterion is defined by* $E_n = \|x_{n+1} - x_n\|_2 < 10^{-4}$.

We consider four cases as follows:

Case 1: $x_1 = (-2, 1, 0)$, $\sigma = 1$, $\rho = 0.5$, $\mu = 0.6$ *and* $\gamma = 1.5$.

Case 2: $x_1 = (-1, 0, 3)$, $\sigma = 2$, $\rho = 0.6$, $\mu = 0.7$ *and* $\gamma = 0.5$.

Case 3: $x_1 = (-4, 0, 2)$, $\sigma = 3$, $\rho = 0.2$, $\mu = 0.3$ *and* $\gamma = 1.9$.

Case 4: $x_1 = (0, -2, 1)$, $\sigma = 4$, $\rho = 0.9$, $\mu = 0.5$ *and* $\gamma = 0.3$.

Using Algorithm 4.1.1 in Theorem 4.1.5, we obtain the following results:

Table 1: Numerical results of Algorithm 4.1.1.

	Case 1	Case 2	Case 3	Case 4
No. of Iter.	159	101	266	119
cpu (Time)	0.0682	0.0593	0.0814	0.2459

The convergence behavior of the error E_n for each Cases is shown in Figure 1 - 4, respectively.

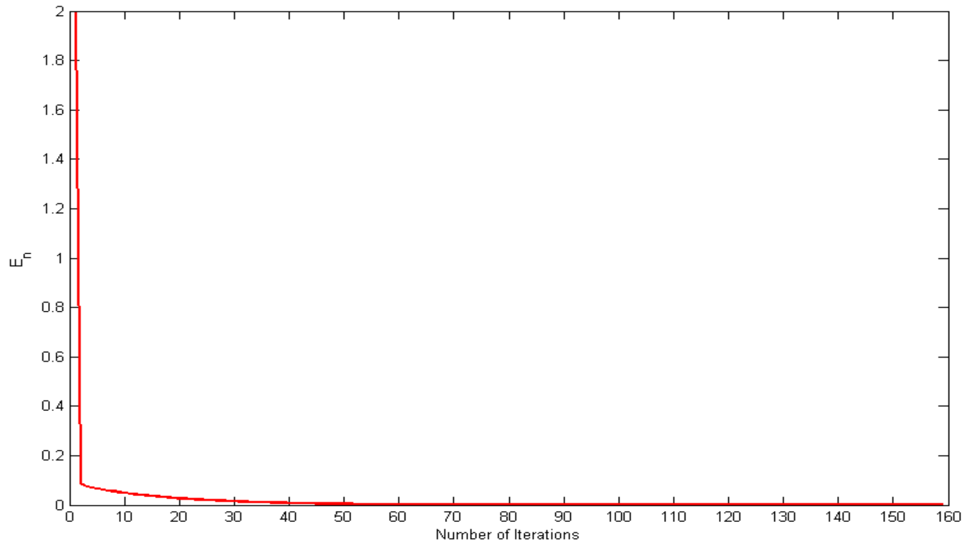

Figure 1: Error plotting E_n for Case 1 in Example 4.3.1

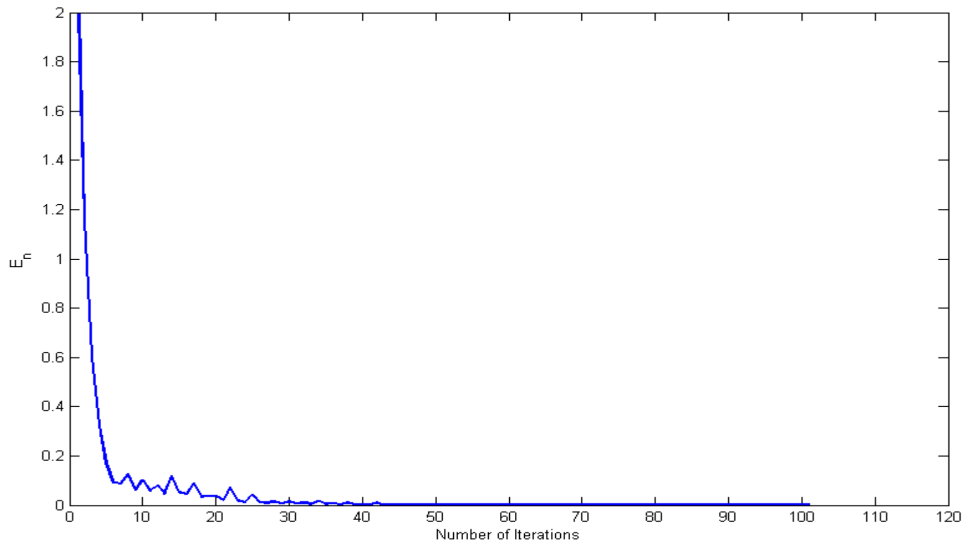

Figure 2: Error plotting E_n for Case 2 in Example 4.3.1

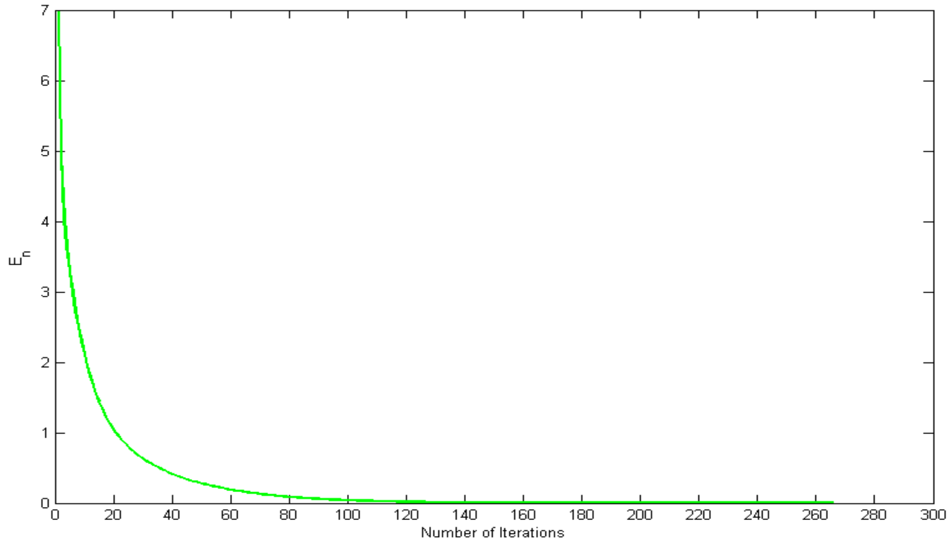

Figure 3: Error plotting E_n for Case 3 in Example 4.3.1

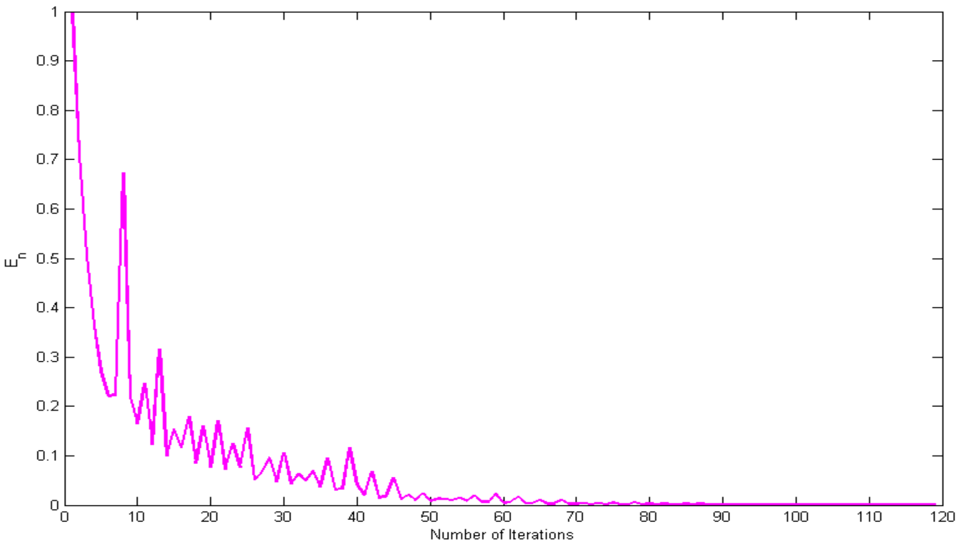

Figure 4: Error plotting E_n for Case 4 in Example 4.3.1

Example 4.3.2. *Consider the following LASSO problem [41]:*

$$\min\{\frac{1}{2}\|Ax-b\|_2^2 : x \in \mathbb{R}^5, \|x\|_1 \leq \tau\}, \tag{4.3.1}$$

where $A = \begin{pmatrix} 1 & 3 & 2 & 1 & 0 \\ 5 & 6 & 1 & -1 & 1 \\ 4 & 2 & 3 & 0 & -2 \\ 0 & 2 & -2 & 1 & 9 \\ 0 & -1 & 3 & 0 & 1 \end{pmatrix}$, $b = (6, 12, 9, 0, 1)$. *We define* $C = \{x \in \mathbb{R}^5 : \|x\|_1 \leq \tau\}$ *and* $Q = \{b\}$. *Since the projection onto the closed convex* C *does not have a closed form solution and so we make use of the subgradient projection. Define a convex function* $c(x) = \|x\|_1 - \tau$ *and denote the level set* C_n *by :*

$$C_n = \{x \in \mathbb{R}^5 : c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0\}, \tag{4.3.2}$$

where $\xi_n \in \partial c(x_n)$. *Then the orthogonal projection onto* C_n *can be calculated by the following:*

$$P_{C_n}(x) = \begin{cases} x, & \text{if } c(x_n) + \langle \xi_n, x - x_n \rangle \leq 0, \\ x - \dfrac{c(x_n) + \langle \xi_n, x - x_n \rangle}{\|\xi_n\|^2}\xi_n, & \text{otherwise.} \end{cases} \tag{4.3.3}$$

It is worth noting that the subdifferential ∂c *at* x_n *is*

$$\partial c(x_n) = \begin{cases} 1, & \text{if } x_n > 0, \\ [-1, 1], & \text{if } x_n = 0, \\ -1, & \text{if } x_n < 0. \end{cases} \tag{4.3.4}$$

The iteration process is stopped when the following criteria satisfied $\|x_{n+1} - x_n\|_2 < 10^{-4}$. *Choose* $\alpha_n = \dfrac{1}{100n}$, *for all* $n \in \mathbb{N}$ *and let* $f(x) = \dfrac{1}{2}x$.

We consider four cases as follows:

Case 1: $x_1 = (-1, 1, 1, 0, 1)$, $\sigma = 1$, $\rho = 0.5$, $\mu = 0.2$ *and* $\gamma = 1.5$.

Case 2: $x_1 = (0, -1, 3, 0, 5)$, $\sigma = 2$, $\rho = 0.4$, $\mu = 0.3$ *and* $\gamma = 0.9$.

Case 3: $x_1 = (1, 9, -2, 0, 5)$, $\sigma = 3$, $\rho = 0.7$, $\mu = 0.6$ *and* $\gamma = 1.9$.

Case 4: $x_1 = (-5, 0, 1, 3, 2)$, $\sigma = 4$, $\rho = 0.2$, $\mu = 0.9$ *and* $\gamma = 0.3$.

Table 2: Numerical results of Algorithm 4.1.6.

	Case 1	Case 2	Case 3	Case 4
No. of Iter.	272	44	228	54
cpu (Time)	0.1444	0.0283	0.2045	0.0204

The convergence behavior of the error E_n for each Cases is shown in Figure 5 - 8, respectively.

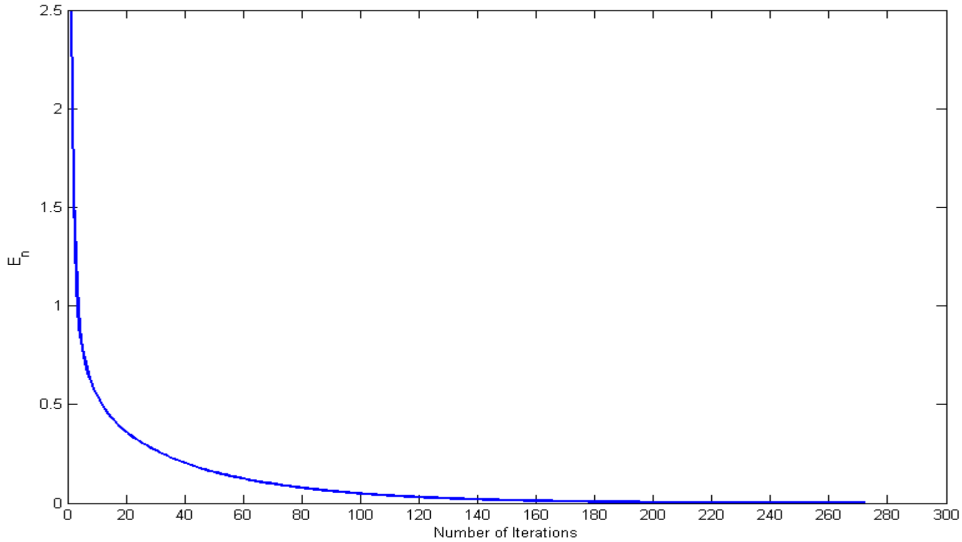

Figure 5: Error plotting E_n for Case 1 in Example 4.3.2

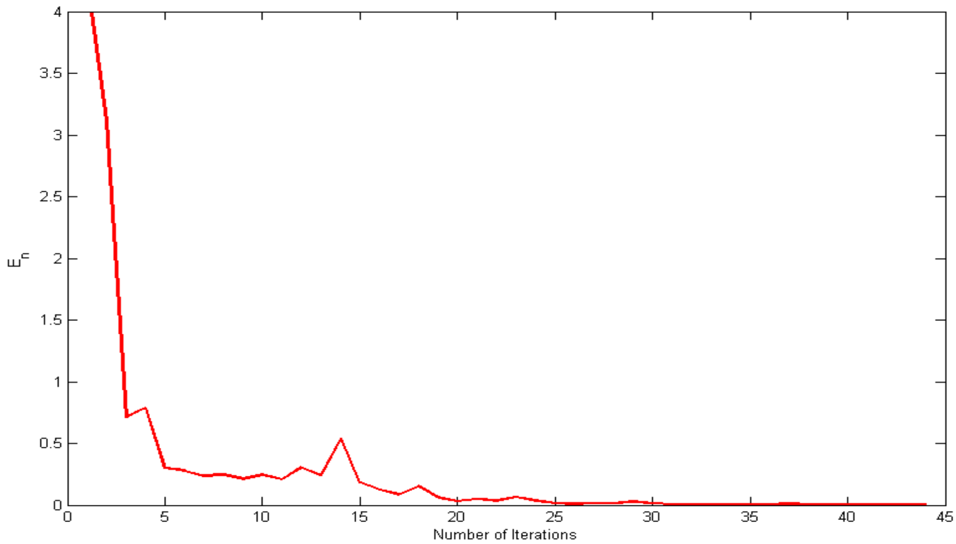

Figure 6: Error plotting E_n for Case 2 in Example 4.3.2

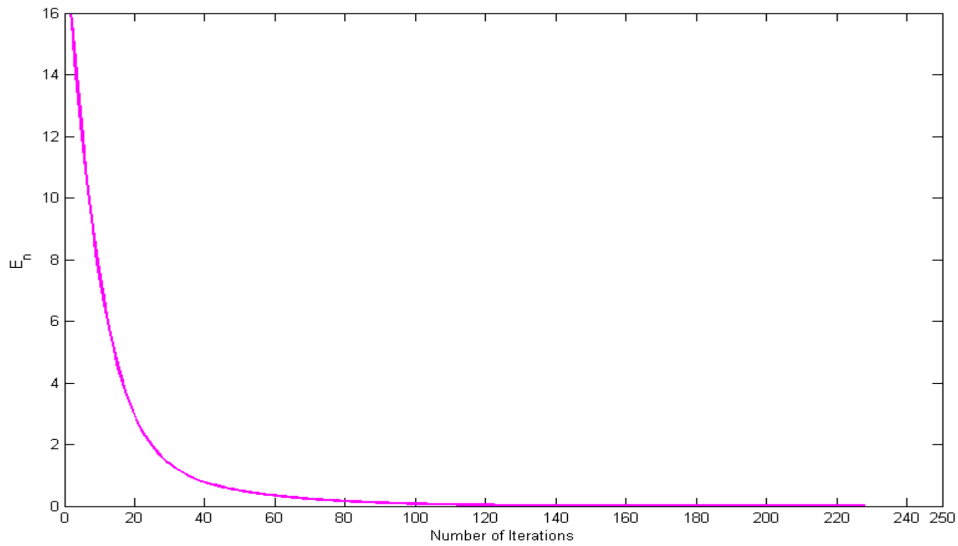

Figure 7: Error plotting E_n for Case 3 in Example 4.3.2

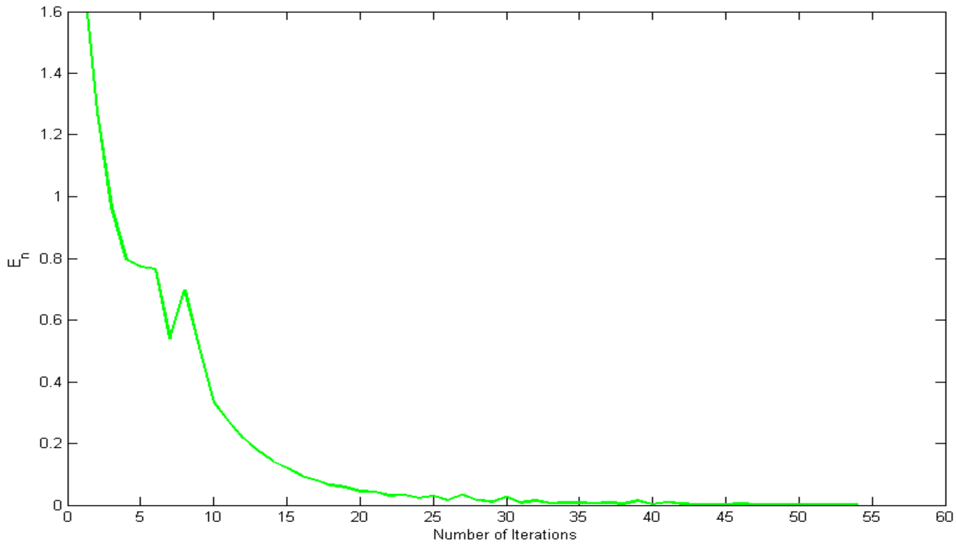

Figure 8: Error plotting E_n for Case 4 in Example 4.3.2

Next, we provide some numerical experiments to the sparse signal recovery in compressed sensing. We illustrate the performance of the relaxed CQ algorithms with the stepsizes defined by Yang [49], López et al. [24], Gibali et al. [16] and our gradient - CQ method (Algorithm 4.2.1). In signal processing, compressed sensing can be modeled as the following under determinated linear equation system:

$$y = Ax + \varepsilon, \tag{4.3.5}$$

where $x \in \mathbb{R}^N$ is a vector with m nonzero components to be recovered, $y \in \mathbb{R}^M$ is the observed or measured data with noisy ε, and $A : \mathbb{R}^N \to \mathbb{R}^M$ $(M < N)$ is a bounded linear observation operator. A is sparse and the range of it is not closed in most inverse problems, thus A is often ill - condition and the problem is also ill - posed. When x is a sparse expansion, finding the solutions of (4.3.5) can be seen as solving the LASSO problem which is the following:

$$\min_{x \in \mathbb{R}^N} \frac{1}{2}\|y - Ax\|_2^2 \text{ subject to } \|x\|_1 \leq t, \tag{4.3.6}$$

where $t > 0$ is a given constant. In particular, if $C = \{x \in \mathbb{R}^N : \|x\|_1 \leq t\}$ and $Q = \{y\}$, then the LASSO problem can be considered as the SFP (2.1.1). From this point of view, we can apply the CQ algorithm to solve (4.3.6).

In our experiment, we test two cases as follow:

Case 1 : $N = 512$, $M = 256$ and $m = 10$;

Case 2 : $N = 4096$, $M = 2048$ and $m = 100$.

The sparse vector $x \in \mathbb{R}^N$ is generated from uniform distribution in the interval $[-2, 2]$ with m nonzero elements. The matrix $A \in \mathbb{R}^{M\times N}$ is generated from a normal distribution with mean zero and one invariance. The observation y is generated by white Gaussian noise with signal - to - noise ratio SNR $= 40$. The process is started with $t = m$ and initial point $x_1 = 0$.

We next give some numerical results by using the CQ algorithms defined by

Yang [49], López et al. [24], Gibali et al. [16] and the gradient - CQ algorithms (Algorithm 4.2.1).

The restoration accuracy is measured by the mean squared error as follows:

$$\mathrm{MSE} = \frac{1}{N}\|x_n - x\|_2^2 < 10^{-5}, \tag{4.3.7}$$

where x_n is an estimated signal of x.

In what follows, let $\tau_n = \frac{1}{\|A\|^2}$ in the CQ algorithm by Yang [49], $\tau_n = \frac{\rho_n\|Ax - y\|^2}{2\|A^*(Ax - y)\|^2}$ with $\rho_n = 3.5$ in (2.1.11) of López et al. [24], τ_n defined by (2.1.15) with $r = 1$, $l = \mu = 0.5$ in that of Gibali et al. [16] and τ_n, φ_n defined by (4.2.3) with $\rho_n = 3.5$, $\theta_n = \frac{1}{(n+1)^5}$. The numerical results are reported as follows.

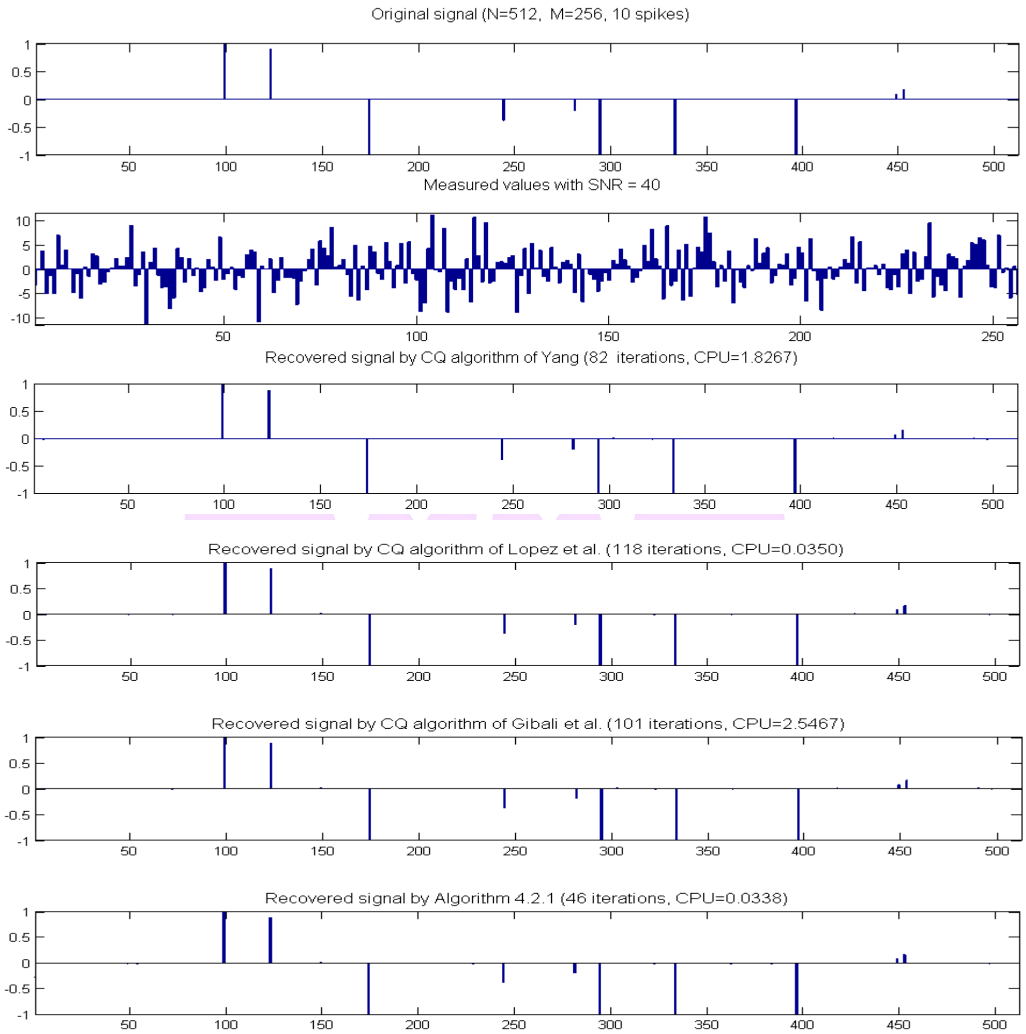

Figure 9: Graph of signal for Algorithm 4.2.1 ($N = 512$, $M = 256$).

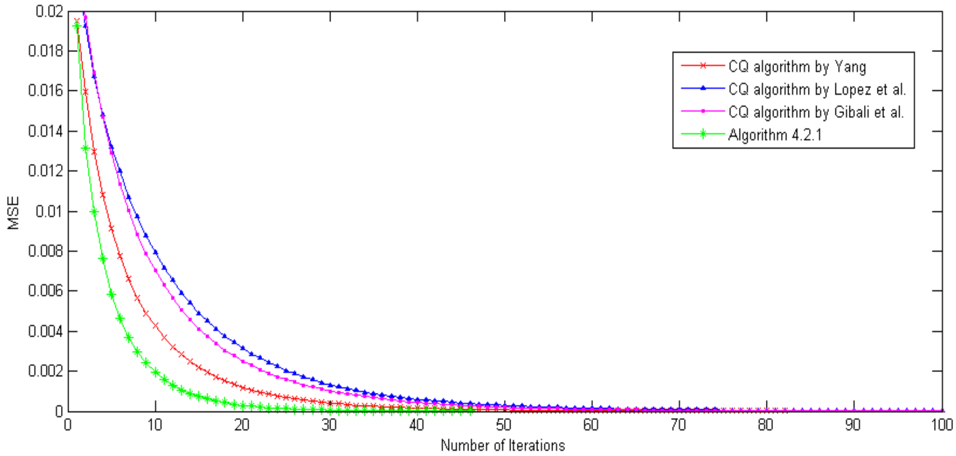

Figure 10: MSE versus iterations for Algorithm 4.2.1 (N = 512, M = 256)

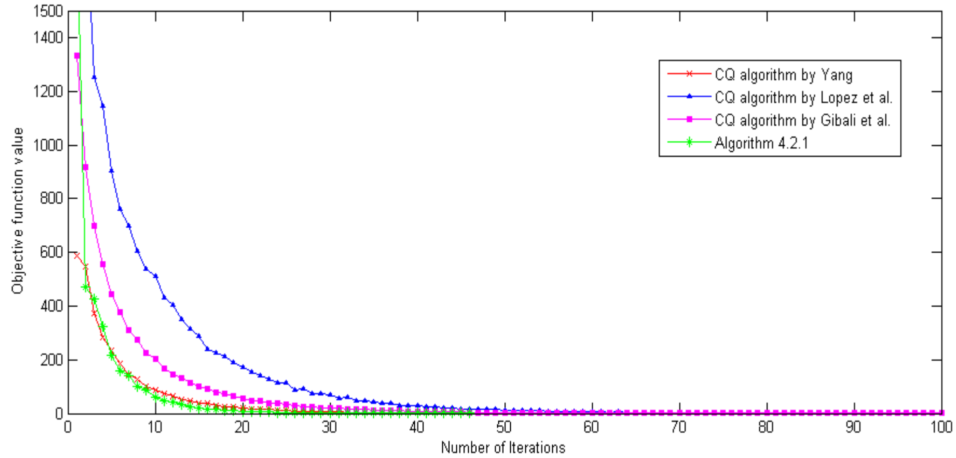

Figure 11: Objective value versus iterations for Algorithm 4.2.1 (N = 512, M = 256)

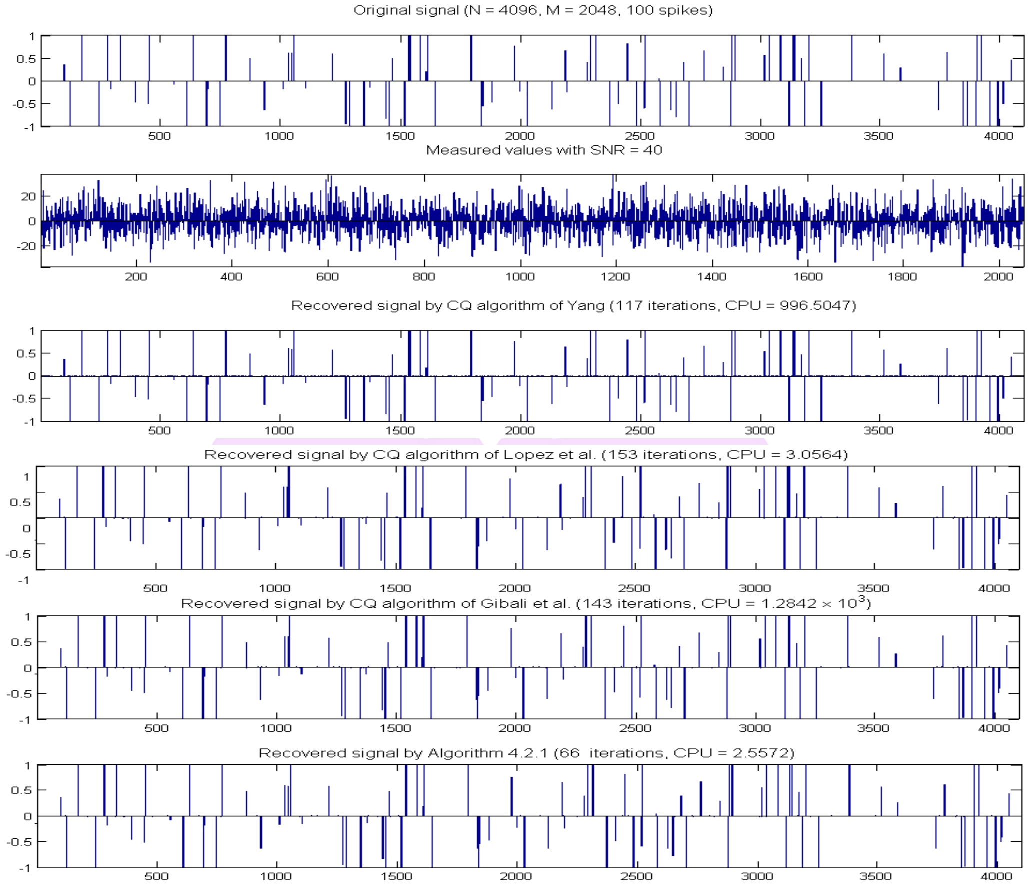

Figure 12: Graph of signal for Algorithm 4.2.1 ($N = 4096$ and $M = 2048$)

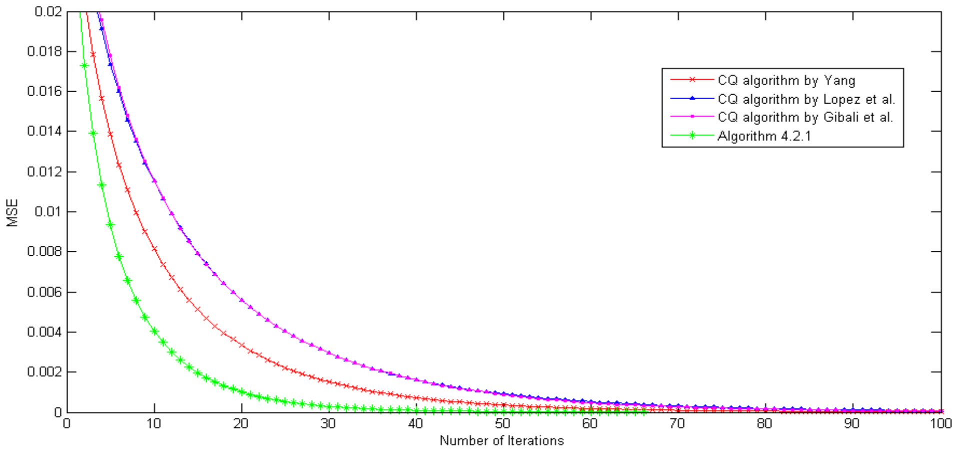

Figure 13: MSE versus iterations for Algorithm 4.2.1 (N = 4096, M = 2048)

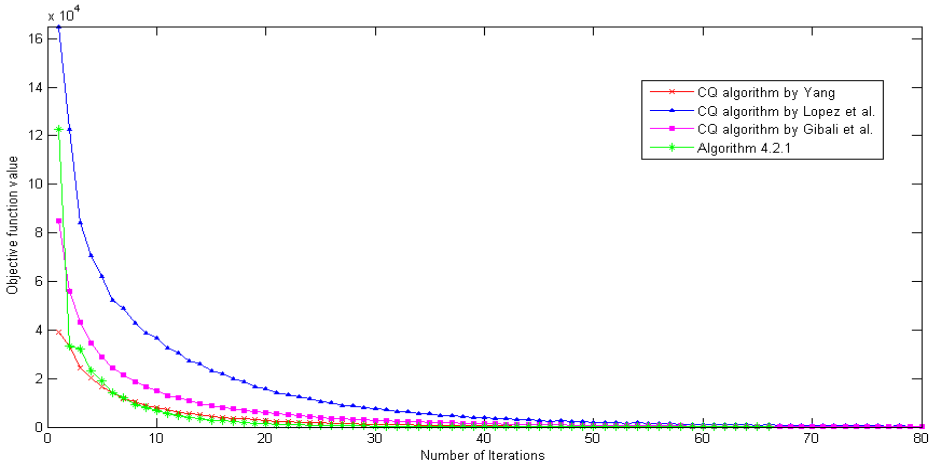

Figure 14: Objective value versus iterations for Algorithm 4.2.1 (N = 4096, M = 2048)

Remark 4.3.3. 1. As shown in Figures 1 and 4, the signal x can be recovered by CQ algorithms. However, it is revealed that among these methods, Algorithm 4.2.1 has the smallest number of iterations and also the shortest cpu time in both Case 1 and Case 2. In the relaxed CQ algorithm by Yang [49], the choice of the stepsize τ_n depends on the operation norm $\|A\|$. So it may take costly time consuming in calculation. In Gibali et al. [16], the stepsize τ_n is computed by Armijo - line search at each iteration. We know that it can be complicated in structure and usually requires many inner iterations to search for a suitable stepsize which may cost a lot of work in cpu time.

2. In Figures 2 and 5, we plot the error value per iterations. It is seen that the errors obtained by Algorithm 4.2.1 decrease faster than those of other CQ algorithms. Also, in Figures 3 and 6, the objective function values obtained by Algorithm 4.2.1 decrease faster than those of other methods.

We finally discuss the strong convergence of the relaxed CQ algorithm (2.1.12) by López et al. [24] and Algorithm 4.2.4.

In this experiment, we set each stepsizes τ_n as in the weak convergence and let $\alpha_n = \dfrac{1}{n+1}$. The initial vector $x_1 = 0$ and u is generated randomly. Then we have the following numerical results.

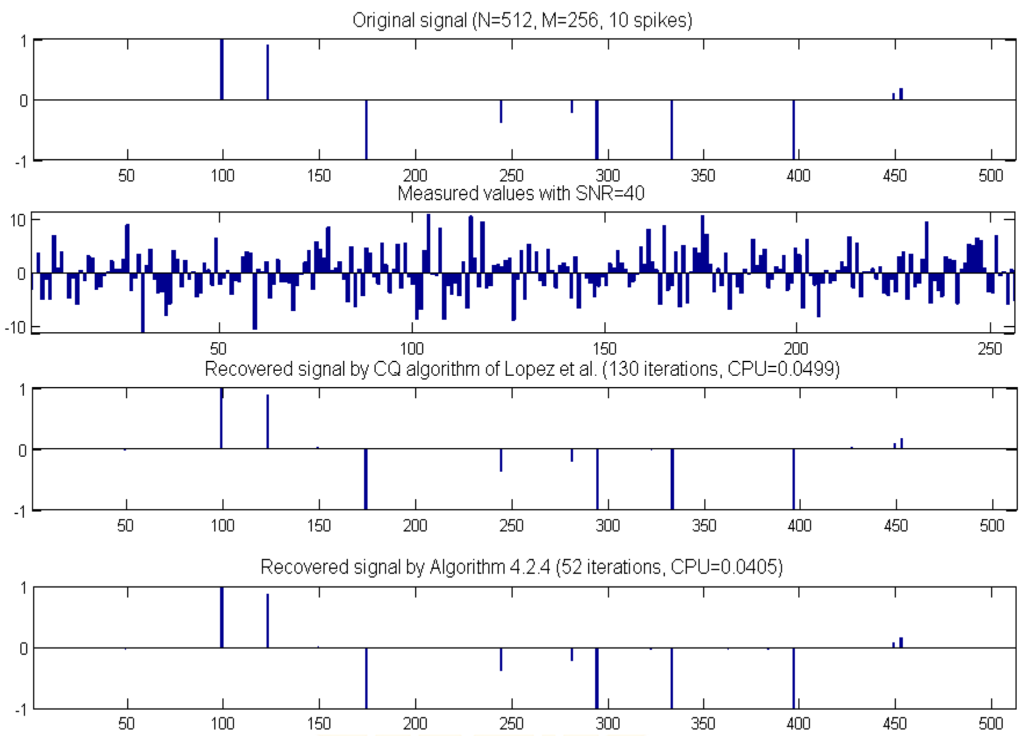

Figure 15: Graph of signal for Algorithm 4.2.4 ($N = 512$, $M = 256$)

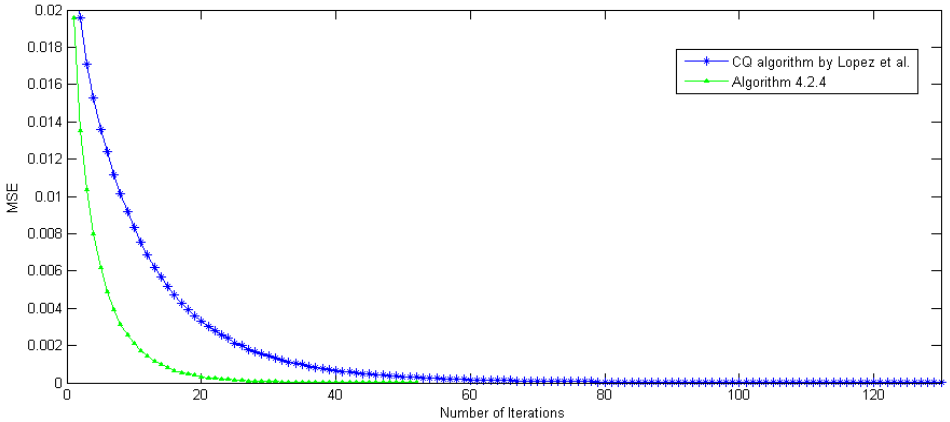

Figure 16: MSE versus iterations for Algorithm 4.2.4 (N = 512, M = 256)

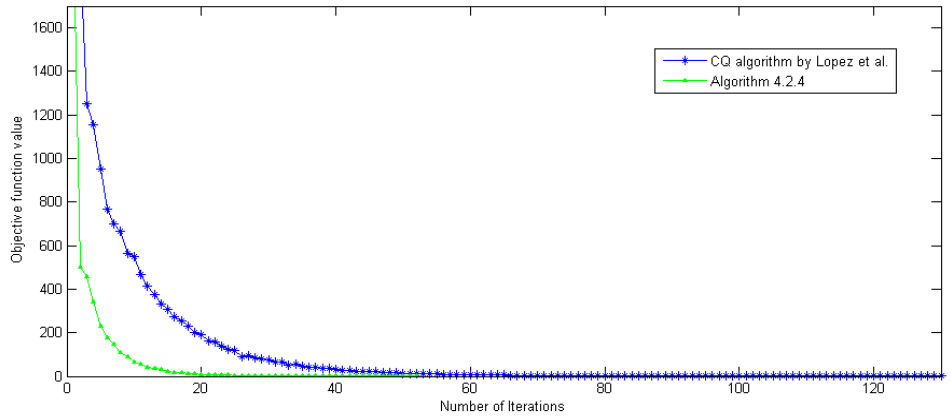

Figure 17: Objective value versus iterations for Algorithm 4.2.4 (N = 512, M = 256)

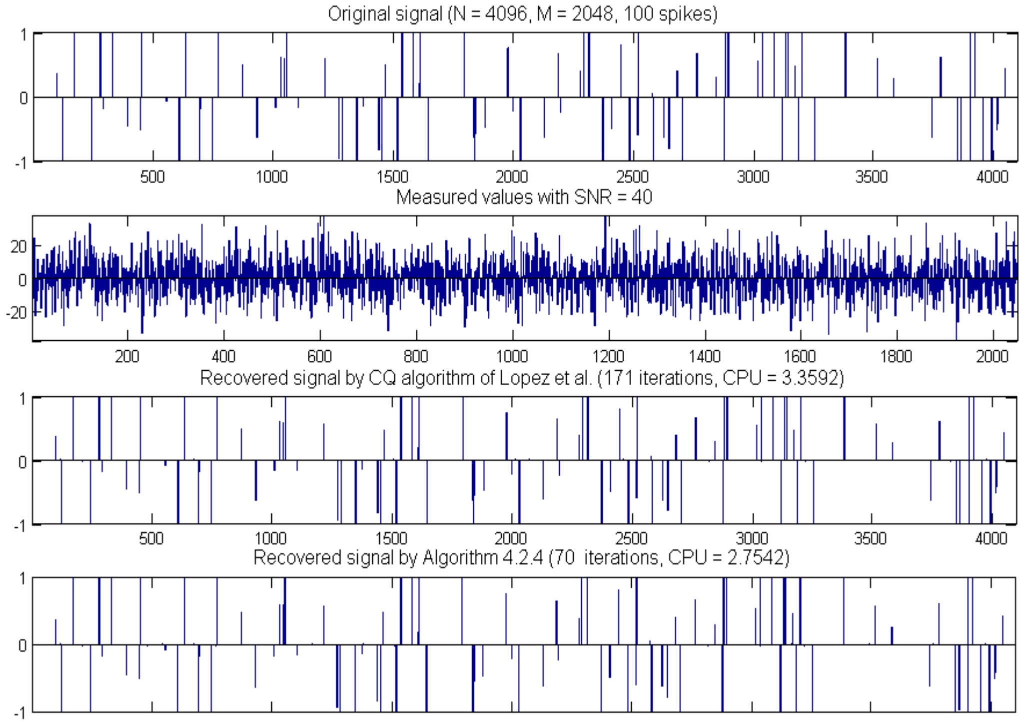

Figure 18: Graph of signal for Algorithm 4.2.4 ($N = 4096$, $M = 2048$)

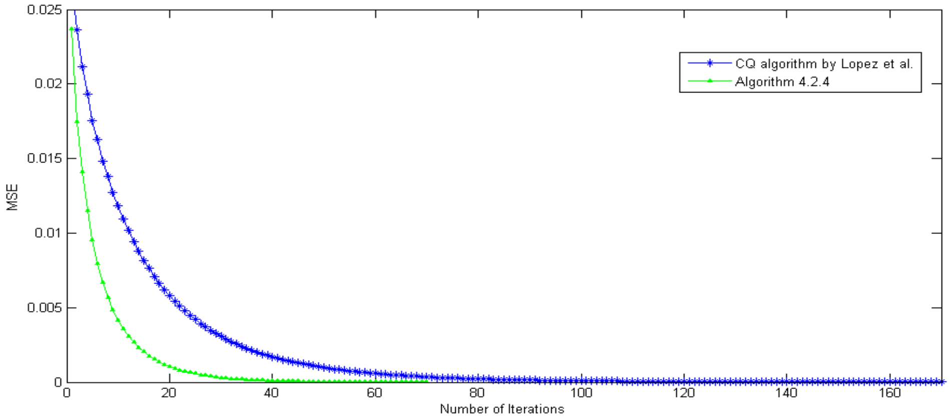

Figure 19: MSE versus iterations for Algorithm 4.2.4 (N = 4096, M = 2048)

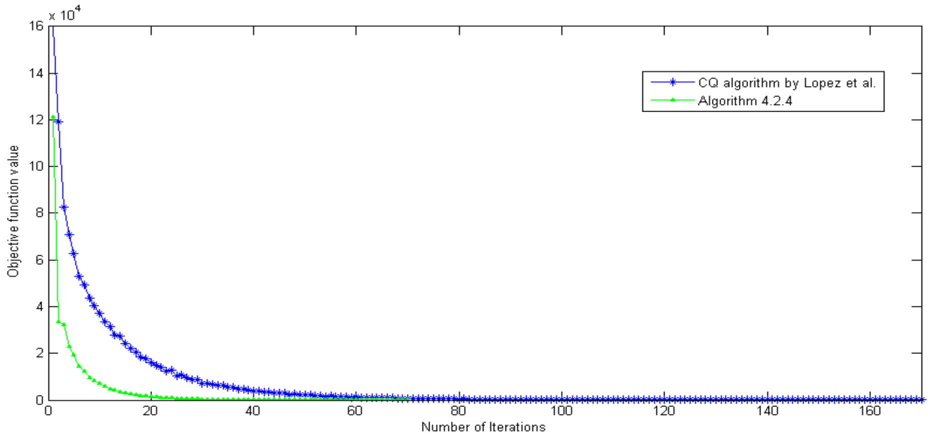

Figure 20: Objective value versus iterations for Algorithm 4.2.4(N = 4096, M = 2048)

Remark 4.3.4. As shown in Figures 7 - 12, the signal x can be recovered with fair degree of accuracy by Algorithm 4.2.4. It appears that our scheme requires the number of iterations and the cpu time less than the Halpern - type relaxed CQ algorithm defined by López et al. [24].

CHAPTER V

CONCLUSION

The following results are all main theorems of this thesis:

Algorithm 4.3.5. *Let $f : H_1 \to H_1$ be a contraction. For any $\sigma > 0, \rho \in (0,1)$ and $\mu \in (0,1)$, choose an arbitrary initial guess $x_1 \in H_1$. Assume x_n and y_n have been constructed. Compute the sequence x_{n+1} via the formula*

$$
\begin{aligned}
y_n &= P_C(x_n - \tau_n F(x_n)) &(4.3.8)\\
x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) &(4.3.9)
\end{aligned}
$$

where $\{\alpha_n\} \subseteq (0,1)$, $\gamma \in (0,2)$ and $\tau_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative integer such that

$$\tau_n \|F(x_n) - F(y_n)\| \leq \mu\|x_n - y_n\|, \tag{4.3.10}$$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n(F(x_n) - F(y_n)) \tag{4.3.11}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n\|(I - P_Q)Ay_n\|^2}{\|d(x_n, y_n)\|^2}. \tag{4.3.12}$$

Theorem 4.3.6. *Assume that $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$. If $S \neq \emptyset$, then the sequence $\{x_n\}$ generated by Algorithm 4.3.5 converges strongly to $z = P_S f(z)$ in S.*

Algorithm 4.3.7. *Let $f : H_1 \to H_1$ be a contraction. For any $\sigma > 0, \rho \in (0,1)$ and $\mu \in (0,1)$, choose an arbitrary initial guess $x_1 \in H_1$. Assume x_n and y_n have been constructed. Compute the sequence x_{n+1} via the formula*

$$
\begin{aligned}
y_n &= P_{C_n}(x_n - \tau_n F_n(x_n))\\
x_{n+1} &= \alpha_n f(x_n) + (1-\alpha_n)(x_n - \gamma\phi_n d(x_n, y_n)) &(4.3.13)
\end{aligned}
$$

where $\{\alpha_n\} \subseteq (0,1)$, $\gamma \in (0,2)$ and $\tau_n = \sigma\rho^{m_n}$ and m_n is the smallest nonnegative *integer such that*

$$\tau_n\|F_n(x_n) - F_n(y_n)\| \le \mu\|x_n - y_n\|, \tag{4.3.14}$$

$$d(x_n, y_n) = (x_n - y_n) - \tau_n(F_n(x_n) - F_n(y_n)) \tag{4.3.15}$$

and

$$\phi_n = \frac{\langle x_n - y_n, d(x_n, y_n)\rangle + \tau_n\|(I - P_{Q_n})Ay_n\|^2}{\|d(x_n, y_n)\|^2}. \tag{4.3.16}$$

Theorem 4.3.8. *Assume that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$. *If* $S \neq \emptyset$, *then the sequence* $\{x_n\}$ *generated by Algorithm 4.3.7 converges strongly to* $z = P_S f(z)$ *in* S.

Algorithm 4.3.9. *Choose an arbitrary initial guess* x_1. *Assume* x_n *and* y_n *have been constructed. Compute* x_{n+1} *via the formula*

$$y_n = x_n - \tau_n F_n(x_n) \tag{4.3.17}$$

$$x_{n+1} = P_{C_n}(y_n - \varphi_n F_n(y_n)) \tag{4.3.18}$$

where F_n *is defined by 4.1.36 and* C_n, Q_n *are given as (4.1.20),*

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|F_n(x_n)\|^2 + \theta_n} \text{ and } \varphi_n = \frac{\rho_n f_n(y_n)}{\|F_n(y_n)\|^2 + \theta_n},\ 0 < \rho_n < 4, 0 < \theta_n < 1. \tag{4.3.19}$$

Theorem 4.3.10. *Assume that* $\inf_n \rho_n(4-\rho_n) > 0$ *and* $\lim_{n\to\infty} \theta_n = 0$. *Then the sequence* $\{x_n\}$ *generated by Algorithm 4.3.9 converges weakly to a point of* S.

Algorithm 4.3.11. *Choose an arbitrary initial guess* x_1. *Assume* x_n *and* y_n *have been constructed. Compute the sequence* x_{n+1} *via the formula*

$$y_n = x_n - \tau_n F_n(x_n) \tag{4.3.20}$$

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)P_{C_n}(y_n - \varphi_n F_n(y_n)) \tag{4.3.21}$$

where $\{\alpha_n\} \subset (0, 1)$, $u \in H_1$, C_n *and* Q_n *are given as (4.1.20),*

$$\tau_n = \frac{\rho_n f_n(x_n)}{\|F_n(x_n)\|^2 + \theta_n} \text{ and } \varphi_n = \frac{\rho_n f_n(y_n)}{\|F_n(y_n)\|^2 + \theta_n}, 0 < \rho_n < 4, 0 < \theta_n < 1. \tag{4.3.22}$$

Theorem 4.3.12. *Assume that* $\{\alpha_n\}$, $\{\rho_n\}$ *and* $\{\theta_n\}$*satisfy the assumptions:*

(a1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*

(a2) $\inf_n \rho_n(4 - \rho_n) > 0$*;*

(a3) $\lim_{n\to\infty} \theta_n = 0$.

Then the sequence $\{x_n\}$ *generated by Algorithm 4.3.11 converges strongly to a point* $P_S u$.

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